Noncommutativity and resolution

(Quantum weighted projective spaces and other examples)

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Aberystwyth, September 2013

References:

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- TB & SAF, Bundles over quantum real weighted projective spaces, *Axioms* 1 (2012), 201–225.
- TB & SAF, Weighted circle actions on the Heegaard quantum sphere, *arXiv:1305.5942*
- TB, Circle actions on a quantum Seifert manifold, *Proceedings of Science (CORFU2011)* 055, 2012.
- TB & A Sitarz, Noncommutative teabags *in preparation*.

Orbifolds

- An orbifold looks locally like a quotient of an open subset of $\mathbb{R}^n$ by a finite group.
- Every n-orbifold is a quotient of a manifold by an almost free action of a subgroup of $O(n)$.
- Every orbifold Riemannian surface is a quotient of a Seifert 3-manifold by an action of S^1.
- Examples:
  - $S^3/\mathbb{Z}_l$;
  - the teardrop manifold: $S^3/U(1)$, $u \triangleright (z_1, z_2) = (uz_1, u^l z_2)$;
  - $\mathbb{T}^2/\mathbb{Z}_2$.

Main objective

To show that deformation of orbifolds might remove singularities.

Levels:

- naïve algebraic (separation of roots);
- homological (finite global dimension);
- differential (freeness of top forms, Poincaré duality);
- advanced algebraic (nondegerate projectively graded algebras);
- topological (change of the C^* description).

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Homological

- An algebra represents a non-singular space if it has a finite global dimension.
- A *projective resolution* of a module M is an exact sequence

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0,$$

  in which the P_i are projective. If, for some n, $P_n \neq 0$ and $P_i = 0$, $i > n$, then n is called the *length* of the resolution.
- *Projective dimension of M*: infimum of lengths of all projective resolutions of M.
- *Global dimension of A,* $\mathrm{gl.dim}\, A$: supremum of projective dimensions of all its modules.

Differential

- A compact orientable manifold has a volume (top) form.
- Algebraically this means that the module of top forms is free (of rank one).

Advanced algebraic

- For a group G, a *G-graded algebra* is an algebra A such that:
  (a) $A = \oplus_{x \in G} A_x$ as a vector space;
  (b) $A_x A_y \subseteq A_{xy}$.
- A G-graded algebra A is said to be *(left) projectively graded* if
  (a) for all $x \in G$, A_x is a finitely generated projective (left) A_e-module;
  (b) $A_x \ncong A_y$ if $x \neq y$;
  (c) *nondegenerate* if A_x is not free for all $x \neq e$.

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Actions, coactions, gradings

- "Action of a (compact abelian) group G on the quantum space X_q" $\equiv$ coaction of the coordinate algebra $\mathcal{O}(G)$ on the coordinate algebra $A = \mathcal{O}(X_q) \equiv$ grading of A by the (discrete) dual group of G.
- E.g.:
  (a) S^1 action of $X_q \equiv \mathbb{Z}$-grading of $A = \mathcal{O}(X_q)$;
  (b) $\mathbb{Z}_p$ action on $X_q \equiv \mathbb{Z}_p$-grading of $A = \mathcal{O}(X_q)$.
- Fixed points of the action $\equiv$ the degree 0 part of A.

Quantum spindles

- $\mathcal{O}(S^3_q)$ is the unital complex $*$-algebra with generators z_0, z_1 subject to the following relations:

$$z_0 z_1 = q z_1 z_0, \qquad z_0 z_1^* = q z_1^* z_0, \quad z_1 z_1^* = z_1^* z_1,$$

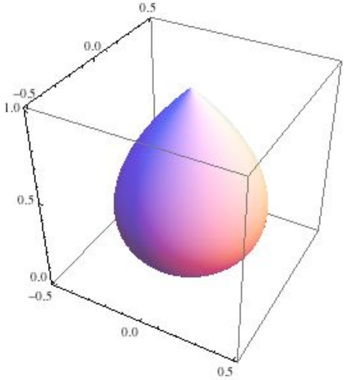
$$z_0 z_0^* = z_0^* z_0 + (q^{-2} - 1) z_1 z_1^*, \quad z_0 z_0^* + z_1 z_1^* = 1.$$

- The action of S^1 on S^3_q, by grading $|z_0| = k$, $|z_1| = l$, for coprime $k, l \in \mathbb{N}$.
- The *quantum spindle* algebra is the degree zero part of $\mathcal{O}(S^3_q)$: it is generated by a, b such that $a^* = a$, $ba = q^{2l} ab$,

$$bb^* = q^{2kl} a^k \prod_{m=0}^{l-1} (1 - q^{2m} a), \quad b^* b = a^k \prod_{m=1}^{l} (1 - q^{-2m} a).$$

Quantum teardrops

$\mathcal{O}(\mathbb{WP}_q(1,l))$ is the *quantum teardrop*, the q-deformation of



Quantum teardrops

- No repeated roots in polynomial relations.
- Global dimension equal to 2. In fact:

Theorem

$\mathcal{O}(\mathbb{WP}_q(k,l))$ is a Noetherian domain of global dimension 2 if $k=1$ and ∞ otherwise.

(this is a consequence of theorems of V Bavula on the structure and dimension of generalized Weyl algebras)

- $C(\mathbb{WP}_q(k,l)) = \mathcal{K}^l \oplus \mathbb{C}$.

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Lens spaces

- $\mathbb{Z}_l$ acts on S^3_q by $|z_0| = 1$, $|z_1| = 0 \pmod{l}$.
- The zero-degree part is the *quantum lens space* $\mathcal{O}(L_q(l;1,l))$, generated by $c = z_0^l$ and $d = z_1$,

$$cd = q^l dc, \qquad cd^* = q^l d^* c, \quad dd^* = d^* d,$$

$$cc^* = \prod_{m=0}^{l-1} (1 - q^{2m} dd^*), \quad c^* c = \prod_{m=1}^{l} (1 - q^{-2m} dd^*),$$

[No repeated roots!]

  - $\mathcal{O}(S^3_q)$ is a projectively $\mathbb{Z}_l$-graded algebra with degree zero part $\mathcal{O}(L_q(l;1,l))$.
  - $\mathcal{O}(L_q(l;1,l))$ is a nondegerate projectively $\mathbb{Z}$-graded algebra with grading $|c| = 1$, $|d| = 1$.
  - $\mathcal{O}(\mathbb{WP}_q(1,l))$ is the degree 0 part of $\mathcal{O}(L_q(l;1,l))$.

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Quantum weighted real projective planes

- The Seifert 3-manifold $\mathcal{O}(\Sigma_q^3)$: generators ζ_0, ζ_1 and the central unitary ξ such that $\zeta_1^* = \zeta_1\xi$

$$\zeta_0\zeta_1 = q\zeta_1\zeta_0, \quad \zeta_0\zeta_0^* = \zeta_0^*\zeta_0 + (q^{-2}-1)\zeta_1^2\xi, \quad \zeta_0\zeta_0^* + \zeta_1^2\xi = 1.$$

- $\mathbb{Z}$-grading: $|\zeta_0| = k$, $|\zeta_1| = l$, $|\xi| = -2l$.
- The $k = 2$, l-odd case $\mathcal{O}(\mathbb{RP}_q^2(l;+))$: selfadjoint a and c_+, $ac_+ = q^{-2l}c_+a$,

$$c_+c_+^* = \prod_{m=0}^{l-1}(1 - q^{2m}a), \quad c_+^*c_+ = \prod_{m=1}^{l}(1 - q^{-2m}a).$$

(a) No repeated roots in polynomial relations.
(b) Global dimension equal to 2.
(c) $C(\mathbb{RP}_q^2(l;+)) = \mathcal{T}_1 \oplus_\sigma \mathcal{T}_2 \oplus_\sigma \cdots \oplus_\sigma \mathcal{T}_l$

Quantum weighted real projective planes

The $k = 1$ case $\mathcal{O}(\mathbb{RP}^2_q(l;-))$: selfadjoint a and b, c_-,

$$ab = q^{-2l}ba, \quad ac_- = q^{-4l}c_-a, \quad b^2 = q^{3l}ac_-, \quad bc_- = q^{-2l}c_-b,$$

$$bb^* = q^{2l}a\prod_{m=0}^{l-1}(1-q^{2m}a), \quad b^*b = a\prod_{m=1}^{l}(1-q^{-2m}a),$$

$$b^*c_- = q^{-l}\prod_{m=1}^{l}(1-q^{-2m}a)b, \quad c_-b^* = q^{l}b\prod_{m=0}^{l-1}(1-q^{2m}a),$$

$$c_-c_-^* = \prod_{m=0}^{2l-1}(1-q^{2m}a), \quad c_-^*c_- = \prod_{m=1}^{2l}(1-q^{-2m}a).$$

(a) No repeated roots in polynomial relations.

(b) Global dimension not known.

(c) $C(\mathbb{RP}^2_q(l;+)) = \underbrace{C(\mathbb{RP}^2_q) \oplus_{\bar{\sigma}} C(\mathbb{RP}^2_q) \oplus_{\bar{\sigma}} \cdots \oplus_{\bar{\sigma}} C(\mathbb{RP}^2_q)}_{l-\text{times}}.$

Noncommutative pillow or teabag

- Noncommutative torus $\mathcal{O}(\mathbb{T}^2_\theta)$: generated by unitary U, V with relation $UV = \lambda VU$, where $\lambda = \exp(2\pi\theta i)$, θ-irrational.
- The $\mathbb{Z}_2$-action:

$$U \mapsto U^*, \qquad V \mapsto V^*.$$

- Invariants $\mathcal{O}(P_\theta)$ generated by $x = U + U^*$, $y = V + V^*$, $z = UV^* + U^*V$ with relations:

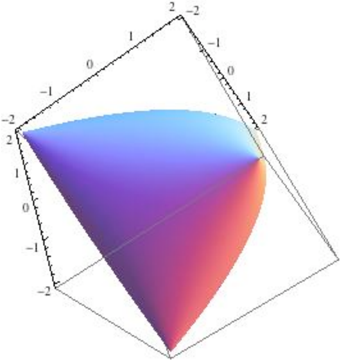
$$yx - \bar{\lambda}xy = \mu z, \quad zx - \lambda xz = -\mu y, \quad yz - \lambda zy = -\mu x,$$

$$x^2 + y^2 + \bar{\lambda}z^2 - xzy = 2(1 + \bar{\lambda}^2),$$

  where $\mu = 2i\sin(2\pi\theta)$.
- $K_0(\mathcal{O}(P_\theta)) = \mathbb{Z}^6$, $K_1(\mathcal{O}(P_\theta)) = 0$.

The pillow

Topologically the deformation of the sphere [Bratteli-Elliott-Evans-Kishimoto], geometrically of the pillow orbifold [Thurston]:



Differential smoothing

- 1-forms on $\mathcal{O}(\mathbb{T}^2_\theta)$: free bimodule generated by $\omega_U = U^* dU$, $\omega_V = V^* dV$, λ-commuting with U, V.
- 2-forms derived from 1-forms: free bimodule generated by a central $\omega = \omega_U \wedge \omega_V$ with algebraic relation

$$\omega_U \wedge \omega_V = -\omega_V \wedge \omega_U.$$

- Differential forms on $\mathcal{O}(P_\theta)$ obtained by considering those on $\mathcal{O}(\mathbb{T}^2_\theta)$ generated by dx, dy and dz.

Differential smoothing

- Relations in 2-forms $\Omega^2(P_\theta)$ on $\mathcal{O}(P_\theta)$:

$$\bar{\lambda} dx \wedge dy + dy \wedge dx = \mu z \omega, \quad \lambda dz \wedge dy + dy \wedge dz = \mu x \omega,$$

$$\lambda dx \wedge dz + dz \wedge dx = -\mu y \omega.$$

- Since $\mu \neq 0$, this means that $x\omega, y\omega, z\omega \in \Omega^2(P_\theta)$.
- Since

$$x^2 + y^2 + \bar{\lambda} z^2 - xzy = 2(1 + \bar{\lambda}^2),$$

  also $\omega \in \Omega^2(P_\theta)$, hence $\Omega^2(P_\theta)$ is a free $\mathcal{O}(P_\theta)$-module isomorphic to $\mathcal{O}(P_\theta)$.

Closing remarks

- Quantization (q-deformation) of orbifolds can resolve their singularities.
- The resolution of singularities can be seen on algebraic, homological and differential (and topological) levels.
- These levels or understanding of resolutions can be both complementary and consistent.