

Ultraproducts, QWEP von Neumann Algebras, and Effros-Maréchal Topology

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Joint work with Uffe Haagerup and Carl Winsløw
(University of Copenhagen)

Outline of Talk

- 1 Kirchberg's QWEP Conjecture
- 2 Effros-Maréchal Topology
- 3 Ultraproduct of von Neumann algebras
- 4 Characterizations of QWEP von Neumann Algebras
 - H. Ando, U. Haagerup, "Ultraproucts of von Neumann algebras", arXiv:1212.5457
 - H. Ando, U. Haagerup, C. Winsløw, "Ultraproducts, QWEP von Neumann algebras, and the Effros-Maréchal topology", arXiv:1306.0460

Kirchberg ('93) revealed remarkable connections among

- Tensor products of C^* -algebras
- Lance's Weak Expectation Property (WEP)
- Connes's Embedding Conjecture (CEC): $\forall N$ sep. II_1 factor embeds into R^ω ?

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In this talk, we discuss how QWEP property is connected to ultraproducts of von Neumann algebras using **topological** method.

QWEP Conjecture

Definition (Lance '73, Kirchberg '93)

- (1) C^* -alg \mathcal{A} has the **weak expectation property** (WEP) if for any faithful representation $\mathcal{A} \subset \mathbb{B}(\mathcal{H})$, there is a ucp map $\Phi: \mathbb{B}(\mathcal{H}) \rightarrow \mathcal{A}^{**}$ s.t. $\Phi|_{\mathcal{A}} = \text{id}_{\mathcal{A}}$.
- (2) C^* -alg \mathcal{A} has the **quotient weak expectation property** (QWEP) if it is the quotient of a C^* -algebra with WEP.

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Theorem (Kirchberg's QWEP Conjecture)

TFAE.

- (1) $C^*(\mathbb{F}_\infty) \otimes_{\min} C^*(\mathbb{F}_\infty) = C^*(\mathbb{F}_\infty) \otimes_{\max} C^*(\mathbb{F}_\infty)$.
- (2) Every C^* -algebra has QWEP.
- (3) $C^*(\mathbb{F}_\infty)$ has WEP.
- (4) (Connes's Embedding Conjecture) Every separable type II_1 factor M admits an embedding into R^ω , where R is the hyperfinite II_1 factor.

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Is QWEP conjecture true?

$$C^*(\mathbb{F}_\infty) \otimes_{\min} C^*(\mathbb{F}_\infty) \stackrel{?}{=} C^*(\mathbb{F}_\infty) \otimes_{\max} C^*(\mathbb{F}_\infty)$$

Kirchberg('93) proved

$$C^*(\mathbb{F}_\infty) \otimes_{\min} \mathbb{B}(\ell^2) = C^*(\mathbb{F}_\infty) \otimes_{\max} \mathbb{B}(\ell^2).$$

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Theorem (Junge-Pisier '95)

$$\mathbb{B}(\ell^2) \otimes_{\min} \mathbb{B}(\ell^2) \neq \mathbb{B}(\ell^2) \otimes_{\max} \mathbb{B}(\ell^2).$$

Fix $H \cong \ell^2$. $\text{vN}(H)$ = set of all vNas on H .

Effros-Maréchal Topology

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- Effros ('65) introduced **Effros Borel structure** on $\text{vN}(H)$.
- Maréchal ('73) introduced **Polish** topology on $\text{vN}(H)$ that generates Effros Borel structure.
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Definition (Maréchal '73)

The **Effros-Maréchal Topology** on $\text{vN}(H)$ is the weakest topology which makes all the maps of the form

$$\text{vN}(H) \ni M \mapsto \|\varphi|_M\|, \quad \varphi \in \mathbb{B}(H)_*$$

continuous.

Definition (Haagerup-Winsløw '98)

For $\{M_n\}_{n=1}^{\infty} \subset \text{vN}(H)$, define $\limsup_{n \rightarrow \infty} M_n$ and $\liminf_{n \rightarrow \infty} M_n$ by

- (1) $\liminf_{n \rightarrow \infty} M_n$ is the set of all $x \in \mathbb{B}(H)$ for which $\exists (x_n)_n \in \ell^\infty(\mathbb{N}, M_n)$ s.t. $x_n \rightarrow x$ *-strongly.
- (2) $\limsup_{n \rightarrow \infty} M_n$ is the vNa generated by those $x \in \mathbb{B}(H)$ for which $\exists (x_n)_n \in \ell^\infty(\mathbb{N}, M_n)$ such that x is a weak-limit point of $\{x_n\}_{n=1}^{\infty}$.

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Theorem (Haagerup-Winsløw '98)

TFAE.

- (1) $M_n \rightarrow M$ in $vN(H)$.
- (2) $\liminf_{n \rightarrow \infty} M_n = M = \limsup_{n \rightarrow \infty} M_n$.

Moreover, $\left(\limsup_{n \rightarrow \infty} M_n \right)' = \liminf_{n \rightarrow \infty} M_n'$ holds.

Important subsets of $\text{vN}(\mathbf{H})$: \mathcal{F} : factors, \mathcal{F}_{inj} injective factors, $\text{vN}(\mathbf{H})^{\text{st}}$ standardly acting vNas, $\mathcal{F}_{\text{II}_1}$ type II_1 factors, etc.

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Theorem (Haagerup-Winsløw '00)

<i>Subset of $vN(H)$</i>	<i>Dense in $vN(H)$?</i>	<i>G_δ?</i>
\mathcal{F}	Yes	Yes
$\bigcup_{n \leq n_0} \mathcal{F}_{\text{I}_n}, n_0 \in \mathbb{N}$	No	Yes (closed)
$\mathcal{F}_{\text{I}_{\text{fin}}}$	*	No (but F_σ)
$\mathcal{F}_{\text{I}_\infty}$	*	No (but F_σ)
$\mathcal{F}_{\text{II}_1}$	Yes	No
$\mathcal{F}_{\text{II}_\infty}$	Yes	No
$\mathcal{F}_{\text{III}_0}$	Yes	No
$\mathcal{F}_{\text{III}_\lambda}, \lambda \in (0, 1)$	Yes	No
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\mathcal{F}_{inj}	*	Yes
\mathcal{F}^{st}	Yes	Yes

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\mathcal{F}_{inj}	*	Yes
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Moreover, * are all equivalent to QWEP (Connes Embedding) conjecture.

Theorem (Haagerup-Winsløw '00)

Let M be a II_1 factor on H . TFAE.

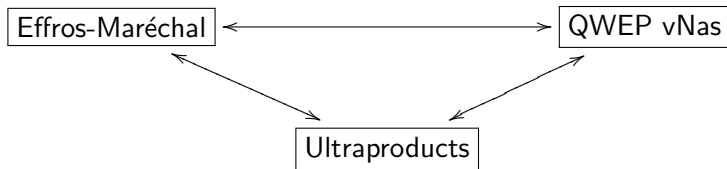
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- (2) There is an embedding $i : M \rightarrow R^\omega$.

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Goal of today's talk: further investigation of $\overline{\mathcal{F}_{\text{inj}}}$ and study the connection of the following:



The Ocneanu Ultraproduct

$(M_n, \varphi_n)_{n=1}^{\infty}$ sequence of vNas/n.f.states. $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$.

Definition (Ocneanu '85)

Define **Ocneanu ultraproduct**

$(M_n, \varphi_n)^\omega := \mathcal{M}^\omega(M_n, \varphi_n) / \mathcal{I}_\omega(M_n, \varphi_n)$, where

$$\mathcal{I}_\omega(M_n, \varphi_n) := \{(x_n)_n \in \ell^\infty(\mathbb{N}, M_n); \|x_n\|_{\varphi_n}^\# \xrightarrow{n \rightarrow \omega} 0\},$$

$$\mathcal{M}^\omega(M_n, \varphi_n) := \{x \in \ell^\infty(\mathbb{N}, M_n); x\mathcal{I}_\omega + \mathcal{I}_\omega x \subset \mathcal{I}_\omega\}.$$

$\varphi^\omega : (x_n)^\omega \mapsto \lim_\omega \varphi_n(x_n)$ is a n.f. state on $(M_n, \varphi_n)^\omega$.

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Remark

Ocneanu considered the case $M_n \equiv M, \varphi_n \equiv \varphi$. In this case $(M, \varphi)^\omega$ is independent of φ , so denote it as M^ω .

However, it is crucial to use the sequence $\{\varphi_n\}_{n=1}^{\infty}$ to study Effros-Maréchal topology.

The Groh-Raynaud Ultraproduct

Taking Ocneanu UP does not commute with taking NC L^p -spaces:
for $p = 1$, $\mathbb{B}(\ell^2)^\omega = \mathbb{B}(\ell^2)$, but $(\mathbb{B}(\ell^2)_*)_\omega \neq \mathbb{B}(\ell^2)_*$

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Is there a right UP $M \mapsto \widetilde{M}^\omega$ s.t. $(L^p(M))_\omega = L^p(\widetilde{M}^\omega)$?

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Theorem (Groh'86, Raynaud'02)

YES! (Groh). More precisely (Raynaud), for each $\{M_n\}_{n=1}^\infty$, $\exists \prod^\omega M_n$
s.t.

- $L^p(\prod^\omega M_n) = (L^p(M_n))_\omega$ ($1 \leq p < \infty$).
In particular, $(\prod^\omega M_n)_* = ((M_n)_*)_\omega$.
- $(\prod^\omega M_n)' = \prod^\omega M_n'$ (in standard form)

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M^ω and $\prod^\omega M$ are VERY different: in fact (Raynaud) $\prod^\omega \mathbb{B}(\ell^2)$ is not semifinite. Moreover (AH'12), while \mathbf{R}^ω is a II_1 factor, $\prod^\omega \mathbf{R}$ is neither semifinite nor a factor!

Ocneanu vs Groh-Raynaud

$M^\omega \neq \prod^\omega M$ in many ways. But are they unrelated?

Consider $\{M_n, \varphi_n\}_{n=1}^\infty$. Then

$$\varphi_\omega = (\varphi_n)_\omega \in ((M_n)_*)_\omega = (\prod^\omega M_n)_*.$$

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$$(1) \quad \sigma_t^{(\varphi_n)^\omega}((x_n)^\omega) = (\sigma_t^{\varphi_n}(x_n))^\omega \text{ for } (x_n)^\omega \in (M_n, \varphi_n)^\omega.$$

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- (5) If M is a type III_0 factor, then $\exists \varphi_n$ s.t. $(M, \varphi_n)^\omega$ is *tracial*.
- (6) $M_\omega \neq M' \cap M^\omega$ in general, but $M_\omega = \mathbb{C} \Rightarrow M' \cap M^\omega = \mathbb{C}$ if M_* separable (solution to Ueda's problem).

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Remark

- If $M \overset{\leftarrow \varepsilon}{\hookrightarrow}_i \mathbb{B}(\ell^2)$, then M is **atomic**.
- If $N \overset{\leftarrow \varepsilon}{\hookrightarrow}_i M$ and M is semifinite, then N is also semifinite (Sakai, Tomiyama).

Proposition (A-Haagerup-Winsløw '13)

Suppose $M_n \rightarrow N$ in $\mathbf{vN}(H)$. Then for any n.f. state $\chi \in \mathbb{B}(H)_*$,

$$N \overset{\leftarrow \varepsilon}{\underset{i}{\hookrightarrow}} (M_n, \psi_n)^\omega,$$

where $\psi_n := \chi|_{M_n}$ and moreover $\varphi = (\psi_n)^\omega \circ i$, $\varphi := \chi|_N$ holds.

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Sketch.

(Construction of i) Let $x \in N = \lim M_n$. Then $N = \liminf M_n$. So $\exists (x_n)_n \in \ell^\infty(\mathbb{N}, M_n)$ s.t. $x_n \xrightarrow{\text{so}^*} x$. Then $(x_n)_n \in \mathcal{M}^\omega(M_n, \psi_n)$ and $i: N \ni x \mapsto (x_n)^\omega \in (M_n, \psi_n)^\omega$ is well-defined.

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where $\psi_n := \chi|_{M_n}$ and moreover $\varphi = (\psi_n)^\omega \circ i$, $\varphi := \chi|_N$ holds.

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(Construction of i) Let $x \in N = \lim M_n$. Then $N = \liminf M_n$. So $\exists (x_n)_n \in \ell^\infty(\mathbb{N}, M_n)$ s.t. $x_n \xrightarrow{\text{so}^*} x$. Then $(x_n)_n \in \mathcal{M}^\omega(M_n, \psi_n)$ and $i: N \ni x \mapsto (x_n)^\omega \in (M_n, \psi_n)^\omega$ is well-defined.

(Construction of ε) Given $y = (y_n)^\omega \in (M_n, \psi_n)^\omega$, let $x := \text{wo-lim}_\omega x_n$. Then as $N = \limsup M_n$, $x \in N$ and $\varepsilon: (M_n, \psi_n)^\omega \ni y \mapsto i(x) \in i(N)$ is well-defined.

By direct calculations, we have $N \overset{\leftarrow \varepsilon}{\underset{i}{\hookrightarrow}} (M_n, \psi_n)^\omega$. □

Corollary (A-Haagerup-Winsløw '13)

Let $M, N \in \mathbf{vN}(H)$ and assume $M_n \cong M$ ($\forall n$) and $M_n \xrightarrow{n \rightarrow \infty} N$ in $\mathbf{vN}(H)$. Then $\exists \{\psi_n\}_{n=1}^{\infty}$ s.t.

$$N \xleftrightarrow[i]{\leftarrow \varepsilon} (M, \psi_n)^{\omega}$$

As a (partial) converse:

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As a (partial) converse:

Proposition (A-Haagerup-Winsløw '13)

Suppose $M_n, N \in \mathbf{vN}(H)^{\text{st}}$ and $\exists \psi_n \in S_{\text{nf}}(M_n)$ s.t.

$N \xleftrightarrow{i}^{\leftarrow \varepsilon} (M_n, \psi_n)^\omega$. Then $\exists u_n \in \mathcal{U}(H)$ and $\exists n_1 < n_2 < \dots$ s.t.

$$u_{n_k} M_{n_k} u_{n_k}^* \xrightarrow{k \rightarrow \infty} N \quad \text{in } \mathbf{vN}(H).$$

With additional efforts, we get the main result:

Theorem (A-Haagerup-Winsløw '13)

For $M \in \mathbf{vN}(H)$, and $0 < \lambda < 1$. TFAE.

- (1) $M \in \overline{\mathcal{F}_{\text{inj}}}$.
- (2) M has QWEP.
- (3) $M \xrightarrow{\leftarrow \varepsilon}_i R_\infty^\omega$. R_∞ : hyperfinite III_1 factor.
- (4) $M \xrightarrow{\leftarrow \varepsilon}_i R_\lambda^\omega$. R_λ : hyperfinite III_λ factor.
- (5) $M \xrightarrow{\leftarrow \varepsilon}_i (M_{k_n}(\mathbb{C}), \varphi_n)^\omega$ for some $\{k_n\}_{n=1}^\infty$ and $\varphi_n \in \mathcal{S}_{\text{nf}}(M_{k_n}(\mathbb{C}))$.
- (6) $\forall \varepsilon > 0, \forall n \in \mathbb{N}, \forall \xi_1, \dots, \xi_n \in \mathcal{P}_M^{\natural}, \exists k \in \mathbb{N}$ and $\exists a_1, \dots, a_n \in M_k(\mathbb{C})_+$ s.t.

$$|\langle \xi_i, \xi_j \rangle - \text{tr}_k(a_i a_j)| < \varepsilon \quad (1 \leq i, j \leq n).$$

Here, \mathcal{P}_M^{\natural} is the natural cone in the standard form of M .

Some consequences:

Corollary

If QWEP conjecture fails, then $\mathfrak{vN}(H)^{-\text{QWEP}} = \{M; M \neq \text{QWEP}\}$ is *open dense* in $\mathfrak{vN}(H)$.

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Theorem (Farah-Hart-Sherman '11)

$\exists M \in \mathcal{F}_{\text{II}_1}$ s.t. $N \hookrightarrow M^\omega$ for every $N \in \mathcal{F}_{\text{II}_1}$.

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Theorem (A-Haagerup-Winsløw'13)

$\exists M \in \mathcal{F}_{\text{III}_1}$ s.t. $N \overset{\leftarrow \varepsilon}{\hookrightarrow}_i M^\omega$ for every $N \in \mathfrak{vN}(H)$.

Proof.

Choose a sequence $\{M_n\}_{n=1}^\infty$ of III_1 factors which is dense in $vN(H)$ and $\varphi_n \in \mathcal{S}_{\text{nf}}(M_n)$. Put $(M, \varphi) := \bigotimes_{\mathbb{N}} (M_n, \varphi_n)$. Then $\forall N$, $\exists \{n_k\}_{k=1}^\infty$ s.t. $M_{n_k} \rightarrow N$. Then as $(M_{n_k}, \varphi_{n_k}) \xrightarrow{i_k^{\varepsilon_k}} (M, \varphi)$ for each k , by our approximation theorem+Connes-Størmer '78 yields $\exists \psi_{n_k} \in \mathcal{S}_{\text{nf}}(M_{n_k})$ s.t.

$$N \xrightarrow{i_0^{\varepsilon_0}} (M_{n_k}, \psi_{n_k})^\omega \cong (M_{n_k}, \varphi_{n_k})^\omega \xrightarrow{(i_k^{\varepsilon_k})^\omega} (M, \varphi)^\omega \cong M^\omega.$$



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Thank you for your attention!