MA10510 & MT10510 – Algebra

Examples of Mathematical Proofs

Example 1 (Direct proof)**.** Prove that the square of an odd integer is odd.

Proof. Let n be an odd integer. Then $n = 2m+1$ for some integer m. Therefore $n^2 = (2m+1)^2 = 4m^2+4m+1 = 2k + 1$ where $k = 2m^2 + 2m$ is an integer. Hence, n^2 is odd. $\square$

Example 2 (Proof by construction)**.** Prove that the following statement is false:

$$\text{all real solutions of } (x-1)^3(x^2+2x+1)(x+1) = 0 \text{ are positive.}$$

Proof. We can provide a counterexample to disprove the statement: note that -1 is a real solution to the equation but -1 is not positive. $\square$

Notice that we actually proved the negation of the statement is true:

$$\text{there exists a non-positive real solution of } (x-1)^3(x^2+2x+1)(x+1) = 0.$$

Given that the negation is true, the original statement is false.

Example 3 (Proof by contraposition)**.** Let $n \in \mathbb{Z}$. Prove that if n^2 is even then n is even.

Proof. Given that $n \in \mathbb{Z}$, we will prove the contrapositive: if n is not even then n^2 is not even. Now an integer is either even or odd. Therefore if n is not even it must be odd. By Example 1 we have that n^2 is also odd. Thus n^2 is not even. We have therefore proved the contrapositive, so the original statement must be true. $\square$

Example 4 (Proof by contradiction)**.** There is no rational number x such that $x^2 = 2$.

Proof. Suppose that there exists a rational number x such that $x^2 = 2$. We may write $x = \frac{p}{q}$ for some integers p, q that have no common factors (other than -1 or 1). Therefore $2 = x^2 = \frac{p^2}{q^2}$ so that $p^2 = 2q^2$, i.e., p^2 is even. By Example 3 we see that p must also be even. So we may write $p = 2m$ for some integer m. It follows that $4m^2 = p^2 = 2q^2$ which implies that q^2 is even. Thus p and q have a common factor of 2, which contradicts our earlier claim that p and q have no common factors (other than -1 or 1). That claim was deduced (correctly) from our assumption that there exists a rational number whose square is 2. It follows that the assumption must be false. So there is no rational number x such that $x^2 = 2$. $\square$

Example 5. Proof by contradiction There are infinitely many prime numbers.

Proof. Suppose the contrary, i.e., that the number of primes is finite. Then we may list them as p_1, p_2‘$, \ldots p_n$ say. Conider the number $n := p_1p_2\cdots p_n + 1$. Given that every positive integer greater than 1 is a product of primes (see Mathematical Induction sheet), we must have that p_i is a factor of n for some $i \in \{1, 2, \ldots, n\}$. As p_i is also a factor of $p_1p_2\cdots p_n$ we have that p_i is a factor of $n - p_1p_2\cdots p_n = 1$, which contradicts our assumption that p_i is a prime. Hence our assumption that the number of primes is finite is false. Therefore, there are infinitely many prime numbers. $\square$