## Extra example: reflection principle for the wave equation.

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, & x>0, \quad t>0 \\ u(x,0) = xe^{-x}, & x>0 \\ u_t(x,0) = 3x^2, & x>0 \\ u(0,t) = 0 & t>0 \end{cases}$$

### The quick way

We derived in the lectures that

$$u(x,t) = \begin{cases} \frac{1}{2}\{f(x+ct) + f(x-ct)\} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\lambda)\,d\lambda \\ \frac{1}{2}\{f(x+ct) - f(ct-x)\} + \frac{1}{2c}\int_{ct-x}^{x+ct} g(\lambda)\,d\lambda, \end{cases}$$

so we could just calculate these, giving

$$u(x,t) = \begin{cases} \frac{1}{2}\{(x+ct)e^{-(x+ct)} + (x-ct)e^{-(x-ct)}\} + \frac{1}{2c}\left((x+ct)^3 - (x-ct)^3\right), & x>ct \\ \frac{1}{2}\{(x+ct)e^{-(x+ct)} - (ct-x)e^{-(ct-x)}\} + \frac{1}{2c}\left((x+ct)^3 - (ct-x)^3\right), & 0<x<ct \end{cases}$$

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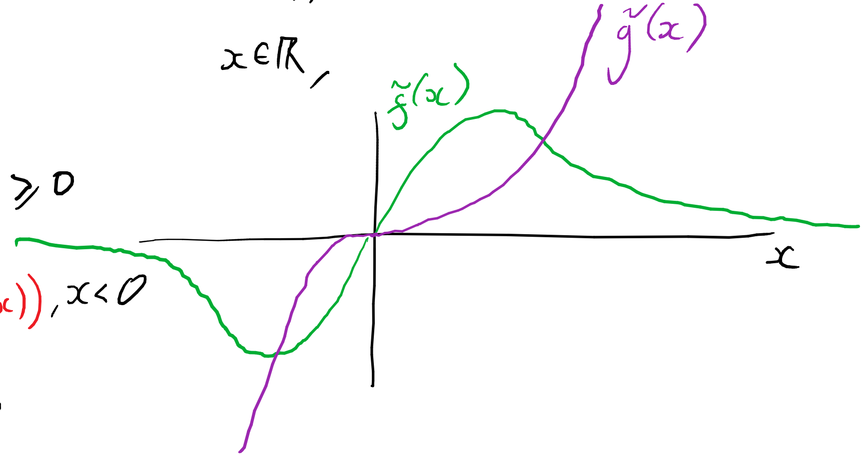
However, if you want to see how it's derived, see overleaf.

We consider the problem (defined on the whole real line, $x \in \mathbb{R}$):

$$(*)\begin{cases} \tilde{u}_{tt} - c^2 \tilde{u}_{xx} = 0, & x \in \mathbb{R}, \quad t > 0 \\ \tilde{u}(x,0) = \tilde{f}(x), & x \in \mathbb{R}, \\ \tilde{u}_t(x,0) = \tilde{g}(x), & x \in \mathbb{R}, \end{cases}$$




where $\tilde{f}(x) = \begin{cases} x e^{-x}, & x \geq 0 \\ x e^{x} \; (= -f(-x)), & x < 0 \end{cases}$

odd extension of $f$

and $\tilde{g}(x) = \begin{cases} 3x^2, & x \geq 0 \\ -3x^2, & x < 0 \end{cases}$

$(*)$ has solution by d'Alembert's formula:

$$\tilde{u}(x,t) = \frac{1}{2}\left\{\tilde{f}(x+ct) + \tilde{f}(x-ct)\right\} + \frac{1}{2c}\int_{x-ct}^{x+ct} \tilde{g}(\lambda)\,d\lambda \qquad (\dagger)$$

Writing in terms of original functions, we have to remember that $\tilde{f}$ and $\tilde{g}$ are defined differently for positive and negative arguments.

When $x > ct$, both $x+ct$ and $x-ct$ are positive, so $\tilde{f} = f$ and $\tilde{g} = g$, hence the "usual looking" d'Alembert's formula on page 1 for $x > ct$.

However, when $0 < x < ct$, $x+ct$ is positive, BUT $x-ct$ is negative.

Thus the integral in $(\dagger)$ becomes trickier, because the definition of the integrand changes somewhere between $x-ct$ and $x+ct$.

So for $0 < x < ct$:

$$\tilde{u}(x,t) = \frac{1}{2}\left\{(x+ct)e^{-(x+ct)} + (x-ct)e^{(x-ct)}\right\} + \frac{1}{2c}\left(\int_{x-ct}^{0} (-3x^2)\,dx + \int_{0}^{x+ct} 3x^2\,dx\right)$$

$$= \frac{1}{2}\left\{(x+ct)e^{-(x+ct)} + (x-ct)e^{(x-ct)}\right\} + \frac{1}{2c}\left(\left[-x^3\right]_{x-ct}^{0} + \left[x^3\right]_{0}^{x+ct}\right)$$

$$= \frac{1}{2}\left\{(x+ct)e^{-(x+ct)} + (x-ct)e^{(x-ct)}\right\} + \frac{1}{2c}\left((x-ct)^3 + (x+ct)^3\right),$$

which is equivalent to the expression for $0 < x < ct$ on page 1.