ADRAN MATHEMATEG / DEPARTMENT OF MATHEMATICS

ARHOLIADAU SEMESTER 1 / SEMESTER 1 EXAMINATIONS

IONAWR / JANUARY 2023

MA34110 - Partial Differential Equations

The questions on this paper are written in English.

Amser a ganiateir - 2 awr

- Gellir rhoi cynnig ar bob cwestiwn.
- Rhoddir mwy o ystyriaeth i berfformiad yn rhan B wrth bennu marc dosbarth cyntaf.
- Cyfrifianellau Casio FX83 neu FX85 YN UNIG a ganiateir.
- Mae modd i fyfyrwyr gyflwyno atebion i'r papur hwn naill ai yn y Gymraeg neu'r Saesneg.

Time allowed - 2 hours

- All questions may be attempted.
- Performance in section B will be given greater consideration in assigning a first class mark.
- Casio FX83 or FX85 calculators ONLY may be used.
- Students may submit answers to this paper in either Welsh or English.

Section A

1. (a) For each of the following equations, state the order and whether it is nonlinear, linear inhomogeneous, or linear homogeneous:

(i) $u_x + 2u_{xy} + 3u_{xxx} = 0$;

(ii) $u_{xx} - 5u_{yy} = x^2 \cos y$;

(iii) $(u_x)^2 + (u_y)^3 = 2e^x$;

(iv) $u_{xx} + 2u_x u_y - u_{yy} = 0$. [8 marks]

(b) Classify the following PDEs as parabolic, elliptic or hyperbolic:

(i) $u_{xx} + 4u_{xy} + 4u_{yy} + u_x = 0$;

(ii) $9u_{xx} + 12u_{xy} - 4u_{yy} - 5u_y - u = 0$. [4 marks]

(c) Find the general solution $u = u(x, y)$ of the PDE $u_{xy} = xy$. [3 marks]

2. Using the method of characteristics and clearly explaining the steps in your solutions, solve the following boundary value problems:

(a) $\begin{cases} u_x + 7u_y = 0, \\ u(0, y) = y^2; \end{cases}$ (b) $\begin{cases} u_x + \cos(x) u_y = 0, \\ u(\pi, y) = \sin y. \end{cases}$

[7,9 marks]

3. Consider the Cauchy problem for the inhomogeneous wave equation on an infinite string with normalised wavespeed:

$$\begin{cases} u_{tt} - u_{xx} = h(x, t) & \text{for } x \in \mathbb{R},\ t > 0, \\ u(x, 0) = f(x) & x \in \mathbb{R}, \\ u_t(x, 0) = g(x) & x \in \mathbb{R}. \end{cases} \tag{1}$$

Stating clearly and precisely any results you use, find the solution of problem (1) for all $x \in \mathbb{R}$, $t > 0$, with f, g, and h given by

$$f(x) = 2x^2; \qquad g(x) = e^{x/2}; \qquad h(x, t) = 12xt^2.$$

Simplify your answer as much as possible. [14 marks]

4. Suppose the temperature, $u = u(x,t)$, of a thin rod satisfies the heat equation $u_t = u_{xx}$ on the domain

$$A = \{(x,t) \in \mathbb{R}^2 : x \in (0,2), t \in (0,7)\},$$

and the following initial and boundary conditions:

$$\begin{cases} u(x,0) = 2x - x^2, & 0 \le x \le 2; \\ u(0,t) = 0, & 0 < t \le 7; \\ u(2,t) = 8t/(1+t), & 0 < t \le 7. \end{cases}$$

(a) Specifically for the domain A defined above, write down the definition of the *parabolic boundary*, Π. [3 marks]

(b) State the maximum and minimum principles for the heat equation. [2 marks]

(c) Without solving the equation and stating clearly any results you use, find the minimum and maximum temperatures of the rod for all $(x,t) \in A$. [8 marks]

(d) State the values of x and t for which the temperature is maximised. [2 marks]

5. Consider the heat equation on the interval $[0,l]$ with Dirichlet boundary conditions

$$\begin{cases} u_t - c^2 u_{xx} = 0, & 0 < x < l,\ 0 < t < T, \\ u(x,0) = f(x), & 0 < x < l, \\ u(0,t) = u(l,t) = 0, & 0 < t < T. \end{cases} \tag{2}$$

(a) Using the method of separation of variables, find the solution of problem (2) in the form

$$u(x,t) = \sum_{k=1}^{\infty} T_k(t) X_k(x).$$

Calculate X_k and T_k. *In this part of the question, you are* **not** *required to apply the initial condition, so your solution may contain arbitrary constants.* [7 marks]

(b) Solve the heat equation $u_t - u_{xx} = 0$ in $0 < x < \pi$, $0 < t < T$, subject to the following boundary and initial conditions:

$$\begin{aligned} u(x,0) &= 4\sin(4x) + 5\sin(9x), \quad 0 \le x \le \pi, \\ u(0,t) &= u(\pi,t) = 0, \qquad\qquad\qquad t \ge 0. \end{aligned}$$

[3 marks]

Section B

6. Explaining your working throughout, use the method of characteristics to find the solution of the following boundary value problem:

$$\begin{cases} u - 2u_x + yu_y = 1, \\ u(0,y) = 1 + y. \end{cases}$$

Give your solution in as simple a form as possible. [12 marks]

7. Consider the inhomogeneous wave equation problem for $u(x,t)$ with homogeneous initial conditions:

$$\begin{cases} u_{tt} - c^2 u_{xx} = h(x,t), & x \in \mathbb{R},\ t > 0, \\ u(x,0) = 0,\ u_t(x,0) = 0, & x \in \mathbb{R}. \end{cases} \tag{3}$$

and also the following problem for $w(x,t,\tau)$:

$$\begin{cases} w_{tt} - c^2 w_{xx} = 0, & x \in \mathbb{R},\ 0 < \tau < t, \\ w(x,\tau,\tau) = 0,\ w_t(x,\tau,\tau) = h(x,\tau), & x \in \mathbb{R}. \end{cases} \tag{4}$$

(a) Prove that if $w(x,t,\tau)$ is a solution of problem (4), then

$$u(x,t) := \int_0^t w(x,t,\tau)\mathrm{d}\tau$$

is a solution of (3). You may use the following Lemma (Leibniz integral rule) without proof:

Lemma 1 *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be smooth. Let $F(t) = \int_0^t f(t,\tau)\mathrm{d}\tau$. Then*

$$F'(t) = f(t,t) + \int_0^t f_t(t,\tau)\mathrm{d}\tau.$$

[6 marks]

(b) Use a suitable change of variables and d'Alembert's formula to solve problem (4). [5 marks]

(c) Hence derive the solution to problem (3). [2 marks]

8. Clearly stating any results you use, find the bounded solution $u = u(x,y)$ of the following boundary value problem:

$$\begin{cases} u_{xx} + u_{yy} = 0, & x > 0,\ y \in \mathbb{R}, \\ u(0,y) = e^{-|y|}. \end{cases}$$

You may leave your answer in the form of an unevaluated integral. [13 marks]

9. Find a solution to the Cauchy problem for the inhomogeneous heat equation with constant dissipation:

$$u_t - u_{xx} + 2u = e^{-2t}, \qquad t > 0,\ x \in \mathbb{R},$$

with the initial condition $u(x,0) = (x+1)^2$.

Hint: make use of the substition of the form $u(x,t) = e^{\alpha t} v(x,t)$ for some choice of $\alpha \in \mathbb{R}$. [12 marks]