MA34110 (&MT34110) PARTIAL DIFFERENTIAL EQUATIONS

S: Seen similar (levels of similarity differ).

U: Unseen.

B: Bookwork (not too much given open-book nature of exam.

Q1 a) First order, linear homogeneous 2

S b) Second order, non-linear 2

c) Third order, linear inhomogeneous 2

d) Second order, linear homogeneous, 2 parabolic 2

10 Q1

Q2 a) Rewriting as $(9,2)\cdot\nabla u = 0$, we see that the directional

S derivative of u in the direction $(9,2)$ is zero, so u is constant along lines in this direction. Such lines are defined by $\frac{dy}{dx} = \frac{2}{9} \Rightarrow y = \frac{2}{9}x + c$. 3

It follows that $u = f(c)$, where $f:\mathbb{R}\to\mathbb{R}$ is an arbitrary function, so $u = f\left(y - \frac{2}{9}x\right)$. Finally, $u(0,y) = \sin y = f(y)$, so the particular solution is $\boxed{u(x,y) = \sin\left(y - \frac{2}{9}x\right)}$ 3

b) Rewriting as $\left(e^{3x}, -\frac{1}{3}\right)\cdot\nabla u = 0$, we see that the directional derivative of u is zero along curves with tangents in the direction $\left(e^{3x}, -\frac{1}{3}\right)$, so u is constant along such curves, which are defined by $\frac{dy}{dx} = -\frac{1}{3}e^{-3x} \Rightarrow y = \frac{1}{9}e^{-3x} + c$. 4

It follows that $u = f(c)$, where $f:\mathbb{R}\to\mathbb{R}$ is an arbitrary function, so $u = f\left(y - \frac{1}{9}e^{-3x}\right)$. Finally, $u(0,y) = \cos y = f\left(y - \frac{1}{9}\right)$, so the particular solution is $\boxed{u(x,y) = \cos\left(y - \frac{1}{9}e^{-3x} + \frac{1}{9}\right)}$ 4

14 Q2

PTO

Q3 Let v, w solve the following problems:

S

$$\begin{cases} v_{tt} - 9v_{xx} = 0, & x \in \mathbb{R}, t>0, \\ v(x,0) = x^2, & x \in \mathbb{R}, \\ v_t(x,0) = \cos 2x, & x \in \mathbb{R}. \end{cases} \qquad \begin{cases} w_{tt} - 9w_{xx} = x \sin t, & x \in \mathbb{R}, t>0 \\ w(x,0) = 0, & x \in \mathbb{R}, \\ w_t(x,0) = 0, & x \in \mathbb{R}. \end{cases}$$

By linearity of the wave equation, $u = v + w$. 3

v has solution by d'Alembert's formula, w by Duhamel's:

- $$v = \tfrac{1}{2}\{(x+3t)^2 + (x-3t)^2\} + \tfrac{1}{6}\int_{x-3t}^{x+3t} \cos(2\lambda)\, d\lambda$$
$$= \tfrac{1}{2}\{x^2 + \cancel{6tx} + 9t^2 + x^2 - \cancel{6tx} + 9t^2\} + \tfrac{1}{12}\left[\sin(2\lambda)\right]_{x-3t}^{x+3t}$$
$$= x^2 + 9t^2 + \tfrac{1}{12}\left(\sin(2x+6t) - \sin(2x-6t)\right)$$ 3

- $$w = \frac{1}{6}\int_0^t \int_{x-3(t-\tau)}^{x+3(t-\tau)} y \sin\tau \, dy\, d\tau$$
$$= \frac{1}{6}\int_0^t \sin\tau \left(\int_{x-3(t-\tau)}^{x+3(t-\tau)} y \, dy\right) d\tau$$
$$= \frac{1}{12}\int_0^t \sin\tau \left(\cancel{x^2} + 6x(t-\tau) + \cancel{9(t-\tau)^2} - \cancel{x^2} + 6x(t-\tau) - \cancel{9(t-\tau)^2}\right)$$
$$= \int_0^t x(t-\tau)\sin\tau \, d\tau \qquad a'=1 \quad b = -\cos\tau$$
$$= xt\int_0^t \sin\tau\, d\tau - x\int_0^t \underset{a}{\tau} \underset{b'}{\sin\tau}\, d\tau$$
$$= -xt\left[\cos\tau\right]_0^t - x\left(\left[-\tau\cos\tau\right]_0^t + \int_0^t \cos\tau \, d\tau\right)$$
$$= xt(1-\cos t) - x(-t\cos t + \sin t)$$
$$= xt - \cancel{xt\cos t} + \cancel{xt\cos t} - x\sin t$$
$$= x(t - \sin t)$$ 6

Combining the above,

$$\boxed{u(x,t) = x^2 + 9t^2 + \tfrac{1}{12}\left(\sin(2x+6t) - \sin(2x-6t)\right) + x(t - \sin t)}$$ 1

13 Q3

PTO

Q4 a) Let u solve $u_t = u_{xx}$ for $(x,t) \in [0,5] \times [0,10]$.

S

Then the maximum value of u occurs on the parabolic boundary

$$\Pi = \left(\{0\} \times [0,10]\right) \cup \left([0,5] \times \{0\}\right) \cup \left(\{5\} \times [0,10]\right). \quad 2$$

b) One end of the rod is held at a steady $0°C$. 2

c) By the maximum and minimum principles, $\max_A u$ and $\min_A u$ occur on Π.

We therefore seek to minimise and maximise:

① $f(x) = x^3 - 7x^2 + 10x$ for $x \in [0,5]$,

② $g(x) = e^{t/5} - 1$ for $t \in [0,10]$,

(and trivially 0 for $t \in [0,10]$). 2

①: f has local minima/maxima where $f'(x) = 0 \Rightarrow 3x^2 - 14x + 10 = 0$

$$\Rightarrow x = \frac{7}{3} \pm \frac{\sqrt{19}}{3},$$

so candidates for minima/maxima over $[0,5]$ are:

$f(0) = 0$,

$f\left(\frac{7}{3} - \frac{\sqrt{19}}{3}\right) = \frac{2}{27}(19\sqrt{19} - 28) \approx 4.06$,

$f\left(\frac{7}{3} + \frac{\sqrt{19}}{3}\right) = -\frac{2}{27}(19\sqrt{19} + 28) \approx -8.21$,

$f(5) = 0$. 4

②: g is monotone increasing, so candidates for maxima/minima are:

$g(0) = e^0 - 1 = 0$,

$g(10) = e^2 - 1 \approx 6.39$. 3

Combining the above, the minimum/maximum temperatures are respectively $-8.21°C$ and $6.39°C$.

1

d) $\frac{7}{3} + \frac{\sqrt{19}}{3} \approx 3.79$ cm (or 1.21 cm from the other end). 1

15

Q4

Q5. Note that $u(0,y) = \sum_{n=1}^{\infty} A_n \sinh(-n\pi)\sin(n\pi y)$

S

$$= \sum_{n=1}^{\infty} \frac{A_n}{2}\left(e^{-n\pi} - e^{n\pi}\right)\sin(n\pi y)$$

$$= \sum_{n=1}^{\infty} \frac{A_n}{2} e^{-n\pi}\left(1 - e^{2n\pi}\right)\sin(n\pi y)$$

$$= 7e^{-4\pi}\left(1 - e^{8\pi}\right)\sin(4\pi y),$$

and so $A_4 = 14$, with $A_n = 0$ for all $n \in \mathbb{N}\setminus\{4\}$.

The solution is therefore $\boxed{u(x,y) = 14\sinh(4\pi(x-1))\sin(4\pi y)}$.

6 Q5

Q6. a) Since $f \in L^1(\mathbb{R})$, the scaled function $f(ax)$ is also absolutely integrable, so its Fourier transform exists. 2

U/B (but fairly straightforward)

Now, $\mathcal{F}\{f(ax)\} = \int_{-\infty}^{\infty} f(ax)e^{i\xi x}\,dx$.

If $a>0$, then making the change of variable $y = ax$ gives

$$\mathcal{F}\{f(ax)\} = \int_{-\infty}^{\infty} f(y)e^{\frac{i\xi y}{a}}\frac{dy}{a} = \frac{1}{a}\int_{-\infty}^{\infty} f(y)e^{iy\frac{\xi}{a}}\,dy = \frac{1}{a}\hat{f}\left(\frac{\xi}{a}\right). \quad 2$$

If $a<0$, then the same change of variables gives

$$\mathcal{F}\{f(ax)\} = \int_{+\infty}^{-\infty} f(y)e^{\frac{i\xi y}{a}}\frac{dy}{a} = -\frac{1}{a}\int_{-\infty}^{\infty} f(y)e^{iy\frac{\xi}{a}}\,dy = -\frac{1}{a}\hat{f}\left(\frac{\xi}{a}\right).$$

2

Combining the above, for $a \in \mathbb{R}\setminus\{0\}$, $\mathcal{F}\{f(ax)\} = \frac{1}{|a|}\hat{f}\left(\frac{\xi}{a}\right)$ 1

b) (i) $\int_{-\infty}^{\infty} f_1(x)e^{i\xi x}\,dx = \int_{-\infty}^{0} e^{x}e^{i\xi x}\,dx = \int_{-\infty}^{0} e^{x(1+i\xi)}\,dx = \frac{1}{1+i\xi}\left[e^{x(1+i\xi)}\right]_{x=-\infty}^{x=0}$

$$= \frac{1}{1+i\xi} = \boxed{\frac{1-i\xi}{1+\xi^2}}. \quad 3$$

(ii) By the result in (a), since $f_2(x) = f_1(5x)$, $\hat{f}_2(\xi) = \frac{1}{5}\left(\frac{1-\left(\frac{i\xi}{5}\right)}{1+\left(\frac{\xi^2}{25}\right)}\right)$

$$= \boxed{\frac{5-i\xi}{25+\xi^2}} \quad 2$$

12 Q6

Section B

Q7 Define characteristic curves, parameterised by t, by

5

$$\begin{cases} \frac{dx}{dt} = 2, & x(0) = 0 \\ \frac{dy}{dt} = 7x, & y(0) = s \end{cases} \Rightarrow \begin{cases} x = 2t \\ y = 7t^2 + s \end{cases} \Rightarrow \begin{cases} t = \frac{x}{2} \\ s = y - \frac{7x^2}{4}. \end{cases}$$

3

On these curves, the PDE reduces to the ODE

$$\frac{du}{dt} + 4u = e^{-4t}\cos t$$

$\left(\text{Since } \frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} = 2u_x + 7xu_y\right).$ 2

Multiply by an integrating factor e^{4t} to give

$$e^{4t}\frac{du}{dt} + 4e^{4t}u = \cos t$$

$$\Leftrightarrow \frac{d}{dt}\left\{e^{4t}u\right\} = \cos t.$$

Integrate w.r.t. t: $e^{4t}u = \sin t + c(s)$

$$\Rightarrow u = e^{-4t}\left(\sin t + c(s)\right)$$

$$= e^{-2x}\left(\sin\left(\frac{x}{2}\right) + c\left(y - \frac{7x^2}{4}\right)\right).$$ 5

The boundary condition $u(0,y) = \sin y$ implies $c(y) = \sin y$,

so $\boxed{u(x,y) = e^{-2x}\left(\sin\left(\frac{x}{2}\right) + \sin\left(y - \frac{7x^2}{4}\right)\right).}$ 2

12 Q7

Q8 a) Hyperbolic 1

B/U

(only analogous reduction to canonical form for elliptic PDEs previously seen)

b) Since coefficients are non-zero, a_{11} can be taken to be 1 without loss of generality (equivalent to dividing through by a_{11}). 2

(1) can then be rewritten (by completing the square) as

$$\left(\frac{\partial}{\partial x} + a_{12}\frac{\partial}{\partial y}\right)^2 u + \left(a_{22} - a_{12}^2\right)\frac{\partial^2}{\partial y^2} u = 0.$$

$$\Leftrightarrow \left(\frac{\partial}{\partial x} + a_{12}\frac{\partial}{\partial y}\right)^2 u - \left(a_{12}^2 - a_{22}\right)\frac{\partial^2}{\partial y^2} u = 0. \quad (*) \qquad 3$$

Since $a_{12}^2 - a_{22} > 0$, we can define $b = \sqrt{a_{12}^2 - a_{22}} > 0$ and introduce new variables via

$$x = \phi, \qquad y = a_{12}\phi + b\eta. \qquad 2$$

Then by the chain rule, $\frac{\partial}{\partial \phi} = \frac{\partial x}{\partial \phi}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \phi}\frac{\partial}{\partial y}$

$= \frac{\partial}{\partial x} + a_{12}\frac{\partial}{\partial y}$,

and $\frac{\partial}{\partial \eta} = \frac{\partial x}{\partial \eta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \eta}\frac{\partial}{\partial y}$

$= b\frac{\partial}{\partial y}$.

Accordingly, $(*)$ can be written as $\frac{\partial^2 u}{\partial \phi^2} - \frac{\partial^2 u}{\partial \eta^2} = 0$. 4

U

Completely unseen

c) By (b), we can define $\phi = x$, $\eta = \frac{1}{\sqrt{3^2 - 1\times 3}}(y - 3x)$

$= \frac{1}{\sqrt{6}}(y - 3x)$, 3

and by doing so, the PDE reduces to $\frac{\partial^2 u}{\partial \phi^2} - \frac{\partial^2 u}{\partial \eta^2} = 0$. We recognise this as the wave equation ($c=1$, $\phi = x$, $\eta = t$), whose general solution is $u(\phi, \eta) = F(\phi + \eta) + G(\phi - \eta)$. 3

Thus the general solution to the given PDE is

$$\boxed{u(x,y) = F\left(x + \frac{1}{\sqrt{6}}(y-3x)\right) + G\left(x - \frac{1}{\sqrt{6}}(y-3x)\right)},$$

where f, g are arbitrary functions of one variable. 2

20

Q8

Q9 a) By the Fourier derivative theorem, $\mathcal{F}\left\{\frac{\partial u}{\partial x}\right\} = -i\xi\bar{u}$. Applying the same

S

result again, $\mathcal{F}\left\{\frac{\partial^2 u}{\partial x^2}\right\} = (-i\xi)^2\bar{u} = -\xi^2\bar{u}$.

Since F.T.s are w.r.t. x, $\mathcal{F}\left\{\frac{\partial^2 u}{\partial y^2}\right\} = \frac{\partial^2 \bar{u}}{\partial y^2}$.

By linearity of the Fourier transform, taking F.T.s therefore yields

$$\frac{\partial^2 \bar{u}}{\partial y^2} - \xi^2\bar{u} = 0.$$

3

S

b) Solving the above gives $\bar{u} = A(\xi)e^{-|\xi|y} + \underline{B(\xi)e^{+|\xi|y}}$.

(although not examples with Robin B.C.s)

Taking F.T.s of the Robin B.C. gives $(\bar{u}_y - \alpha\bar{u})\big|_{y=0} = \bar{h}(\xi)$

We discard the underlined term because it is unbounded. 2

Applying the B.C. gives $-|\xi|A(\xi) - \alpha A(\xi) = \bar{h}(\xi)$

$$\Rightarrow A(\xi) = -(|\xi|+\alpha)^{-1}\bar{h}(\xi),$$

and so $\boxed{\bar{u}(\xi, y) = -\frac{\bar{h}(\xi)e^{-|\xi|y}}{|\xi|+\alpha}.}$ 4

9 Q9

Q10 From lecture notes, the heat equation IVP on the whole real line

U

has solution

$$u(x,t) = \frac{1}{2c\sqrt{\pi t}}\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4c^2t}} f(y)\,dy.$$

If f is odd, then so is u spatially, since

(since f odd)

$$u(-x,t) = \frac{1}{2c\sqrt{\pi t}}\int_{-\infty}^{\infty} e^{-\frac{(x+y)^2}{4c^2t}} f(y)\,dy = \frac{1}{2c\sqrt{\pi t}}\int_{+\infty}^{-\infty} e^{-\frac{(x-z)^2}{4c^2t}}(-f(z))(-1)\,dz$$

substitution $z=-y$

$$= \frac{-1}{2c\sqrt{\pi t}}\int_{-\infty}^{\infty} e^{-\frac{(x-z)^2}{4c^2t}} f(z)\,dz = -u(x,t).$$

3

We therefore odd-extend the problem to the whole real line

Define $\tilde{u}(x,t) = \begin{cases} u(x,t), & x>0, \\ -u(-x,t), & x<0. \\ 0, & x=0 \end{cases}$ and $\tilde{f}(x,t) = \begin{cases} f(x), & x>0, \\ -f(-x), & x<0, \\ 0, & x=0 \end{cases}$

In doing so, the boundary condition is automatically satisfied, and so

[continues over]

$\tilde{u}$ solves $\begin{cases} \tilde{u}_t - c^2 \tilde{u}_{xx} = 0, & x \in \mathbb{R}, \ t>0 \\ \tilde{u}(x,0) = \tilde{f}(x), & x \in \mathbb{R}. \end{cases}$ ⊛

Its restriction to $x>0$ is the solution we seek. 3

⊛ has solution:

$$\tilde{u}(x,t) = \frac{1}{2c\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4c^2 t}} \tilde{f}(y)\,dy.$$

$$= \frac{1}{2c\sqrt{\pi t}} \left(\int_{-\infty}^{0} e^{-\frac{(x-y)^2}{4c^2 t}} (-f(-y))\,dy + \int_0^\infty e^{-\frac{(x-y)^2}{4c^2 t}} f(y)\,dy \right)$$

$z=-y$

$$= \frac{1}{2c\sqrt{\pi t}} \left(-\int_0^\infty e^{-\frac{(x+z)^2}{4c^2 t}} (-f(z))(-1)\,dz + \int_0^\infty e^{-\frac{(x-y)^2}{4c^2 t}} f(y)\,dy \right)$$

$$= \boxed{\frac{1}{4\sqrt{\pi t}} \int_0^\infty \left(e^{-\frac{(x-y)^2}{16t}} - e^{-\frac{(x+y)^2}{16t}} \right) f(y)\,dy} \quad \text{(since } c=2\text{)}$$

3 9 Q10