PRIFYSGOL ABERYSTWYTH UNIVERSITY

ADRAN MATHEMATEG / DEPARTMENT OF MATHEMATICS

ARHOLIADAU SEMESTER 1 / SEMESTER 1 EXAMINATIONS

IONAWR / JANUARY 2022

MA34110 - Partial Differential Equations

The questions on this paper are written in English.

If you have questions about the paper during the exam, contact the module co-ordinator, Dr Adam Vellender, on asv2@aber.ac.uk.

You should write out solutions to the paper and upload them to Blackboard as a single PDF file.

Amser a ganiateir - 3 awr
Mae'n rhaid cyflwyno eich atebion erbyn 12:30 (amser y DU).

- Gellir rhoi cynnig ar bob cwestiwn.
- Rhoddir mwy o ystyriaeth i berfformiad yn rhan B wrth bennu marc dosbarth cyntaf.
- Mae modd i fyfyrwyr gyflwyno atebion i'r papur hwn naill ai yn y Gymraeg neu'r Saesneg.

Time allowed - 3 hours
Submission must be completed by 12:30 (UK time).

- All questions may be attempted.
- Performance in section B will be given greater consideration in assigning a first class mark.
- Students may submit answers to this paper in either Welsh or English.

Section A

1. For each of the following equations, state the order and whether it is nonlinear, linear inhomogeneous, or linear homogeneous. Where an equation is second order and linear, state further whether it is parabolic, hyperbolic, or elliptic.

(a) $5u_x - 2u_y = 0$, (b) $u_{xx} - 2u_{yy} = u^{1/2}$,

(c) $u_{xxx} + xu_{xy} = 7\sin^2(x - 3y)$, (d) $u_{xx} + 4u_{xy} + 4u_{yy} = 0$.

[10 marks]

2. Using the method of characteristics and clearly explaining the steps in your solutions, solve the following boundary value problems:

(a) $\begin{cases} 9u_x + 2u_y = 0, \\ u(0, y) = \sin y. \end{cases}$ (b) $\begin{cases} e^{3x}u_x = \frac{1}{3}u_y, \\ u(0, y) = \cos y. \end{cases}$

[6,8 marks]

3. Clearly stating any results you use, solve the following Cauchy problem for the one-dimensional inhomogeneous wave equation:

$$\begin{cases} u_{tt} - 9u_{xx} = x\sin t, & x \in \mathbb{R},\ t > 0, \\ u(x, 0) = x^2, & x \in \mathbb{R}, \\ u_t(x, 0) = \cos(2x), & x \in \mathbb{R}. \end{cases}$$

[13 marks]

4. Suppose the temperature (measured in °C), u, of a rod of length 5cm satisfies the heat equation $u_t = u_{xx}$ on the domain

$$A := \{(x, t) \in \mathbb{R}^2 :\ x \in [0, 5],\ t \in [0, 10]\},$$

and the following initial and boundary conditions:

$$\begin{cases} u(x, 0) = x^3 - 7x^2 + 10x, & 0 \le x \le 5, \\ u(0, t) = 0, & 0 \le t \le 10, \\ u(5, t) = \exp(t/5) - 1, & 0 \le t \le 10. \end{cases}$$

Here, x and t are respectively measured in cm and seconds, and $\exp$ denotes the exponential function.

(a) State the maximum principle for the heat equation specifically for this domain. [2 marks]

(b) Give a physical interpretation of the boundary condition $u(0, t) = 0$. [2 marks]

(c) Without solving the equation and stating clearly any results you use, find the minimum and maximum temperatures of the rod for all $(x, t) \in A$, giving your answers to two decimal places. [10 marks]

(d) How far from the end of the rod is the minimum temperature realised? [1 mark]

5. By seeking a separable solution to the following Laplace equation problem on the unit square:

$$\begin{cases} u_{xx} + u_{yy} = 0, & x \in [0,1],\ y \in [0,1], \\ u(x,0) = u(x,1) = 0, & x \in [0,1], \\ u(0,y) = f(y), & y \in [0,1], \\ u(1,y) = 0, & y \in [0,1], \end{cases}$$

it is possible to show that the solution is of the form

$$u(x,y) = \sum_{n=1}^{\infty} A_n \sinh\left(n\pi(x-1)\right) \sin\left(n\pi y\right).$$

If $f(y) = 7e^{-4\pi}(1 - e^{8\pi})\sin(4\pi y)$, find the constants A_n for all $n \in \mathbb{N}$ and hence state the particular solution. [6 marks]

6. Let $f \in L^1(\mathbb{R})$. Then the Fourier transform of f exists and is defined for $\xi \in \mathbb{R}$ as

$$\mathcal{F}\{f(x)\} = \bar{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{i\xi x} \mathrm{d}x.$$

(a) Show that for $a \in \mathbb{R}\backslash\{0\}$, $\mathcal{F}\{f(ax)\} = \frac{1}{|a|}\bar{f}\left(\xi/a\right)$. [7 marks]

(b) Calculate the Fourier transforms of the following two functions:

(i) $f_1(x) = \begin{cases} \exp(x), & x < 0, \\ 0, & x \geq 0. \end{cases}$ (ii) $f_2(x) = \begin{cases} \exp(5x), & x < 0, \\ 0, & x \geq 0. \end{cases}$

[5 marks]

[Section B begins overleaf]

Section B

7. Explaining your working throughout, find the solution of the following boundary value problem:

$$\begin{cases} 2u_x + 7xu_y + 4u = \cos(x/2)e^{-2x}, \\ u(0,y) = \sin y. \end{cases}$$

[12 marks]

8. Consider the second order linear PDE

$$a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} = 0, \tag{1}$$

where a_{ij} are real constants, each non-zero, such that $a_{12}^2 - a_{11}a_{22} > 0$.

(a) Classify (1) as either elliptic, hyperbolic, or parabolic. [1 mark]

(b) By using a suitable change of independent variables (converting x and y to ϕ and η, say), show that (1) can be reduced to the canonical form

$$u_{\phi\phi} - u_{\eta\eta} = 0.$$

[11 marks]

(c) Hence, stating clearly any facts you use, give the general solution of the PDE

$$u_{xx} + 6u_{xy} + 3u_{yy} = 0.$$

[8 marks]

9. Let $u = u(x,y)$ be a bounded solution to the following boundary value problem defined in a half-plane:

$$\begin{cases} u_{xx} + u_{yy} = 0, & x \in \mathbb{R},\ y > 0, \\ (u_y - \alpha u)\big|_{y=0} = h(x), & x \in \mathbb{R}. \end{cases} \tag{2}$$

Here, $h \in L^1(\mathbb{R})$ is a known function, while $\alpha > 0$ is a constant.

Define the Fourier transform with respect to x of u as

$$\mathcal{F}\{u\} = \bar{u}(\xi, y) = \int_{-\infty}^{\infty} u(x,y)e^{i\xi x}\mathrm{d}x.$$

(a) Clearly stating any results you use, show that the Fourier transform with respect to x of u satisfies

$$\frac{\partial^2 \bar{u}}{\partial y^2} - \xi^2 \bar{u} = 0.$$

[3 marks]

(b) Hence find an explicit expression for $\bar{u}(\xi, y)$, the Fourier transform of the solution to problem (2). [6 marks]

10. Stating clearly any results from the lectures that you use, solve the following Cauchy problem for the heat equation on a semi-infinite rod:

$$\begin{cases} u_t - 4u_{xx} = 0, & x > 0,\ t > 0, \\ u(x,0) = f(x), & x > 0, \\ u(0,t) = 0, & t > 0. \end{cases}$$

[9 marks]