MA/MT34110 Partial Differential Equations - January 2021

S: seen similar before
(levels of similarity differ)

U: unseen.

SOLUTIONS

Q1 a) S

(i) Third order, linear, 2

(ii) Second order, linear, 2

(iii) Second order, non-linear, 2

(iv) First order, non-linear. 2

b) (i) $a_{11}a_{22} = 2(-1) = -2 < \frac{1}{4} = \left(\frac{1}{2}\right)^2 = a_{12}^2$, so the PDE is hyperbolic,

(ii) $a_{11}a_{22} = 9 \times 4 = 36 = 6^2 = a_{12}^2$, so the PDE is parabolic. 4

c) $u_{xy} = 8xy$

INTEGRATE W.R.T. y

$\Rightarrow u_x = 4xy^2 + f(x)$

INTEGRATE W.R.T. x

$\Rightarrow u = 2x^2y^2 + f_1(x) + g(y)$,

where f, f_1, g are arbitrary functions of a single variable. 3

(15)

Q2. S

a) Rewrite as $(1,-4)\cdot\nabla u = 0$. That is, the directional derivative of u in the direction $(1,-4)$ is zero, so u is constant in this direction. Lines in the direction $(1,-4)$ are defined by $\frac{dy}{dx} = -4 \Rightarrow y = c - 4x$, $c \in \mathbb{R}$. 4 Since u is constant along each line, it follows that $u(x,y) = f(c) = f(4x+y)$. The boundary condition gives $u(0,y) = f(y) = \cos y$, so $u(x,y) = \cos(4x+y)$. 3

b) Rewrite as $(1,-3x^2)\cdot\nabla u = 0$. That is, the directional derivative of u along curves whose tangents lie in the direction $(1,-3x^2)$ is zero, so u is constant along such curves, which are defined by $\frac{dy}{dx} = -3x^2 \Rightarrow y = c - x^3$, $c \in \mathbb{R}$. 5 Since u is constant along such curve, it follows that $u(x,y) = f(c) = f(x^3+y)$. The boundary condition gives $u(0,y) = f(y) = y^2$, so $u(x,y) = (x^3+y)^2$. 4

(16)

Q3. S

a) $\Pi = \left(\{0\}\times[0,5]\right) \cup \left(\{4\}\times[0,5]\right) \cup \left([0,4]\times\{0\}\right)$. 3

b) Let $u = u(x,t)$ be a solution of the heat equation on $[0,\ell]\times[0,T]$. Then the maximum and minimal values of u lie on the parabolic boundary, Π. 2

c) The maximum and minimum principles yield that u's minimum and maximum values lie on Π; we therefore minimise and maximise u on Π. 2 That is, we minimise and maximise $4\sin\left(\frac{5\pi x}{24}\right)$ for $x \in [0,4]$ and $2e^{-t}$ for $t \in [0,5]$.
The latter, being monotone decreasing, is maximised when $t=0$, where $2e^{-t}\big|_{t=0} = 2$, and minimised at $t=5$, where $2e^{-t}\big|_{t=5} = 2e^{-5}$. 2
The former is maximised over $[0,4]$ where $\frac{5\pi x}{24} = \frac{\pi}{2} \Rightarrow x = \frac{12}{5}$, and $4\sin\left(\frac{5\pi x}{24}\right)\Big|_{x=\frac{12}{5}} = 4$.
Note also that $4\sin\left(\frac{5\pi x}{24}\right)$ is non-negative for all $x \in [0,4]$, taking the value zero when $x=0$.
Combining the above, the minimum and maximum temperatures are 0 and 4 units respectively. 4

d) $t=0$, $x = \frac{12}{5}$. 2

(15)

Q4. a) Wavespeed. 1

S b) Waves propagate at constant speed c. The front of the wave propagating to the right will have position $1+ct$, so will reach $x=5$ when $t=\frac{4}{c}$. The "back" of the wave has position $x=ct$ at time t, so reaches $x=5$ when $t=\frac{5}{c}$.

Thus $u(5,t)$ is non-zero for $\frac{4}{c} \leq t \leq \frac{5}{c}$. 3

c) By linearity of the wave equation, $u=v+w$, where v and w solve

$$\begin{cases} V_{tt} - c^2 V_{xx} = 0, & x\in\mathbb{R}, t>0; \\ V(x,0) = f(x), & x\in\mathbb{R}; \\ V_t(x,0) = g(x), & x\in\mathbb{R}. \end{cases} \qquad \begin{cases} W_{tt} - c^2 W_{xx} = h(x,t), & x\in\mathbb{R}, t>0; \\ W(x,0) = 0, & x\in\mathbb{R}; \\ w_t(x,0) = 0, & x\in\mathbb{R}. \end{cases}$$

4

We find v by employing d'Alembert's formula:

$$V = \tfrac{1}{2}\{f(x+ct) + f(x-ct)\} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\lambda)\,d\lambda$$
$$= \tfrac{1}{2}\{(x+t)^2 + (x-t)^2\} + \tfrac{1}{2}\int_{x-t}^{x+t} \cos(3\lambda)\,d\lambda$$
$$= x^2 + t^2 + \tfrac{1}{6}\Big[\sin(3\lambda)\Big]_{\lambda=x-t}^{\lambda=x+t}$$
$$= x^2 + t^2 + \tfrac{1}{6}\big(\sin(3(x+t)) - \sin(3(x-t))\big)$$
$$= x^2 + t^2 + \tfrac{1}{3}\sin(3t)\cos(3x).$$

4

w can be found by using Duhamel's formula:

$$W = \frac{1}{2c}\int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} h(y,\tau)\,dy\,d\tau$$
$$= \frac{1}{2}\int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} y e^{-\tau}\,dy\,d\tau$$
$$= \frac{1}{4}\int_0^t e^{-\tau}\big[y^2\big]_{x-c(t-\tau)}^{x+c(t-\tau)}\,d\tau$$
$$= \cancel{\tfrac{1}{4}}\int_0^t e^{-\tau}\big(\cancel{4}cx(t-\tau)\big)\,d\tau$$
$$= xt\int_0^t e^{-\tau}\,d\tau - x\int_0^t \tau e^{-\tau}\,d\tau$$
$$= xt\big[-e^{-\tau}\big]_{\tau=0}^{\tau=t} - x\Big(\big[-\tau e^{-\tau}\big]_0^t - \int_0^t e^{-\tau}\,d\tau\Big)$$
$$= xt(1-e^{-t}) - x(-te^{-t} - e^{-t} + 1)$$
$$= x(t - \cancel{te^{-t}} + \cancel{te^{-t}} + e^{-t} - 1)$$
$$= x(t + e^{-t} - 1).$$

4

Combining the above, $u = v+w = x^2 + t^2 + \frac{1}{3}\sin(3t)\cos(3x) + x(t+e^{-t}-1)$. 2

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Q5. Note that
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$$u(0,y) = \sum_{n=1}^{\infty} A_n \sinh(-n\pi)\sin(n\pi y)$$
$$= \sum_{n=1}^{\infty} \frac{A_n}{2}\left(e^{-n\pi} - e^{n\pi}\right)\sin(n\pi y)$$
$$= \sum_{n=1}^{\infty} \frac{A_n}{2} e^{-n\pi}\left(1 - e^{2n\pi}\right)\sin(n\pi y) = e^{-3\pi}\left(1 - e^{6\pi}\right)\sin(3\pi y).$$

3

It follows that $A_3 = 2$ and $A_n = 0$ for all $n \in \mathbb{N}\setminus\{3\}$, hence the solution is

$$u(x,y) = 2\sinh\left(3\pi(x-1)\right)\sin(3\pi y).$$

3 6

SECTION B

Q6. Characteristic curves, parameterised by t, are defined by
S

$$\begin{cases} \dfrac{dx}{dt} = 3, & x(0) = s \\ \dfrac{dy}{dt} = -1, & y(0) = 0 \end{cases} \Rightarrow \begin{cases} x = 3t + s \\ y = -t \end{cases} \Leftrightarrow \begin{cases} s = x + 3y \\ t = -y \end{cases}.$$

4

Along these curves, $\dfrac{du}{dt} = \dfrac{\partial u}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial u}{\partial y}\dfrac{dy}{dt} = 3\dfrac{\partial u}{\partial x} - \dfrac{\partial u}{\partial y}$, so the PDE reduces to the ODE:

$$\frac{du}{dt} - \frac{3}{3t+s}u = 0.$$

2

This is first order and linear, so multiply by an integrating factor $e^{-\int \frac{3}{3t+s}dt} = e^{-\ln(3t+s)} = (3t+s)^{-1}$, to yield

$$(3t+s)^{-1}\frac{du}{dt} - 3(3t+s)^{-2}u = 0$$
$$\Leftrightarrow \frac{d}{dt}\left\{(3t+s)^{-1}u\right\} = 0$$
$$\Rightarrow (3t+s)^{-1}u = c(s) \quad (c \text{ an arbitrary function})$$
$$\Rightarrow u = c(s)(3t+s) = x\,c(x+3y).$$

4

Applying the boundary condition, $u(x,0) = xc(x) = xe^x$, so $c(x) = e^x$. It follows that the particular solution to the PDE is

$$u(x,y) = xe^{x+3y}.$$

2 12

Q7. a) Let $u(x,t) = -2xt - x^2$. Then $u_x(x,t) = -2t - 2x$, so $u_{xx}(x,t) = -2$,

U

and $u_t = -2x$. Thus $u_t - xu_{xx} = -2x - x(-2) = 0$. 3

Now maximise u over $[-2,2]\times[0,1]$. Local critical points satisfy

$u_x = 0 \Rightarrow 2t + 2x = 0 \Rightarrow x = -t$, and

$u_t = 0 \Rightarrow -2x = 0 \Rightarrow x = 0$, and $u(0,0) = 0$. 2

Along boundaries:

$u(-2,t) = 4t - 4$ (max over $(0,1)$ is 0, min -4)

$u(2,t) = -4t - 4$ (max over $(0,1)$ is -4, min -8)

$u(x,0) = -x^2$ (max over $(-2,2)$ is 0, min -4)

$u(x,1) = -2x - x^2$ (max over $(-2,2)$ is $+1$, min -8).

$= -(x+1)^2 + 1$

4

Combining, the maximum value of u is 1, with $u(-1,1) = 1$. 1

b) The proof seen in lectures introduces $v(x,t) = u + \varepsilon x^2$. Then v satisfies $v_t - xv_{xx} = (u_t - xv_{xx}) - 2\varepsilon x = -2\varepsilon x$. In the constant coefficient case, we argued that this quantity was strictly negative. This is not the case for $x < 0$ here.

6 16

Q8. a) Since $f \in L^1(\mathbb{R})$, the Fourier transform exists.

Mostly U (a, b seen similar)

$$\bar{f}(\xi) = \int_{-\infty}^{\infty} e^{-x^2} e^{i\xi x}\,dx = \int_{-\infty}^{\infty} e^{-(x^2 - i\xi x)}\,dx = \int_{-\infty}^{\infty} e^{-((x-\frac{i\xi}{2})^2 + \frac{\xi^2}{4})}\,dx$$

$$= e^{-\xi^2/4}\int_{-\infty}^{\infty} e^{-(x-\frac{i\xi}{2})^2}\,dx \qquad \alpha = x - \frac{i\xi}{2}, \quad d\alpha = dx$$

$$= e^{-\xi^2/4}\int_{-\infty}^{\infty} e^{-\alpha^2}\,d\alpha$$

$$= \sqrt{\pi}\, e^{-\xi^2/4}.$$

5

b) $a_{11}a_{22} = 1\times 1 = 1 > 0 = a_{12}^2$; elliptic. 2

c) By the derivative theorem for F.T.s: $\mathcal{F}\{u_x\} = -i\xi\,\bar{u}(\xi,y)$,

so $\mathcal{F}\{u_{xx}\} = (-i\xi)^2\bar{u}(\xi,y) = -\xi^2\bar{u}(\xi,y)$.

Also, $\mathcal{F}\{u_{yy}\} = \frac{\partial^2\bar{u}}{\partial y^2}$, and $\mathcal{F}\{e^{-x^2}\} = \sqrt{\pi}\,e^{-\xi^2/4}$. Thus taking F.T.s of the given Laplace's equation problem gives:

$$\frac{\partial^2\bar{u}}{\partial y^2} - \xi^2\bar{u} = \sqrt{\pi}\,e^{-\xi^2/4}.$$

4

d) The problem arrived at in (c) is essentially an ODE. It is second order, inhomogeneous, with auxiliary equation $m^2 - \xi^2 = 0$, so the complementary function is $\bar{u}_{comp}(\xi, y) = A(\xi)e^{\xi y} + B(\xi)e^{-\xi y}$.

The forcing term is not a function of y. Seek a particular integral in the form $\bar{u}_{PARTICULAR}(\xi, y) = C(\xi)$. Substituting into the ODE gives

$$-\xi^2 C(\xi) = \sqrt{\pi}\, e^{-\xi^2/4} \Rightarrow C(\xi) = -\frac{\sqrt{\pi}}{\xi^2} e^{-\xi^2/4}.$$

Thus $$\bar{u}(\xi, y) = A(\xi)e^{\xi y} + B(\xi)e^{-\xi y} - \frac{\sqrt{\pi}}{\xi^2} e^{-\xi^2/4}.$$ 4

Apply B.C.s:

$$\bar{u}(\xi, 0) = A(\xi) + B(\xi) - \frac{\sqrt{\pi}}{\xi^2} e^{-\xi^2/4} = 0 \quad (1)$$

$$\bar{u}(\xi, 1) = A(\xi)e^{\xi} + B(\xi)e^{-\xi} - \frac{\sqrt{\pi}}{\xi^2} e^{-\xi^2/4} = 0 \quad (2)$$

From (1), $A(\xi) = \frac{\sqrt{\pi}}{\xi^2} e^{-\xi^2/4} - B(\xi)$. Substitute into (2):

$$\frac{\sqrt{\pi}}{\xi^2} e^{-\xi^2/4 + \xi} - B(\xi)e^{\xi} + B(\xi)e^{-\xi} = \frac{\sqrt{\pi}}{\xi^2} e^{-\xi^2/4}$$

$$\Rightarrow B(\xi) = \frac{\sqrt{\pi}}{\xi^2} e^{-\xi^2/4} \frac{1 - e^{\xi}}{e^{-\xi} - e^{\xi}}.$$

$$\Rightarrow A(\xi) = \frac{\sqrt{\pi}}{\xi^2} e^{-\xi^2/4} \left(1 - \frac{1 - e^{\xi}}{e^{-\xi} - e^{\xi}}\right)$$

$$= \frac{\sqrt{\pi}}{\xi^2} e^{-\xi^2/4} \frac{e^{-\xi} - 1}{e^{-\xi} - e^{\xi}}.$$

Combining, $$\bar{u}(\xi, y) = \frac{\sqrt{\pi}}{\xi^2} e^{-\xi^2/4} \left\{ \frac{e^{-\xi} - 1}{e^{-\xi} - e^{\xi}} e^{\xi y} + \frac{1 - e^{-\xi}}{e^{-\xi} - e^{\xi}} e^{-\xi y} - 1 \right\}.$$ 6

e) $$u(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{u}(\xi, y)\, e^{-i\xi x}\, d\xi.$$ 1

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