PRIFYSGOL ABERYSTWYTH UNIVERSITY

ADRAN MATHEMATEG / DEPARTMENT OF MATHEMATICS

ARHOLIADAU SEMESTER 1 / SEMESTER 1 EXAMINATIONS

IONAWR / JANUARY 2021

MA34110 - Partial Differential Equations

The questions on this paper are written in English.

If you have questions about the paper during the exam, contact the module co-ordinator, Dr Adam Vellender, on asv2@aber.ac.uk.

You should write out solutions to the paper and upload them to Blackboard as a single PDF file.

Amser a ganiateir - 3 awr
Mae'n rhaid cyflwyno eich atebion erbyn 12:30 (amser y DU).

- Gellir rhoi cynnig ar bob cwestiwn.
- Rhoddir mwy o ystyriaeth i berfformiad yn rhan B wrth bennu marc dosbarth cyntaf.
- Mae modd i fyfyrwyr gyflwyno atebion i'r papur hwn naill ai yn y Gymraeg neu'r Saesneg.

Time allowed - 3 hours
Submission must be completed by 12:30 (UK time).

- All questions may be attempted.
- Performance in section B will be given greater consideration in assigning a first class mark.
- Students may submit answers to this paper in either Welsh or English.

Section A

1. (a) For each of the following equations, state the order and whether it is nonlinear, linear inhomogeneous, or linear homogeneous:

(i) $u_x + 3u_{xy} + 4u_{xxx} = 0$;
(ii) $u_{xx} - 2u_{yy} = xy^2$;
(iii) $u_{xx} + 3u_x u_y - u_{yy} = 0$;
(iv) $(u_x)^2 + (u_y)^2 = 2\cos x$. [8 marks]

(b) Classify the following PDEs as parabolic, elliptic or hyperbolic:

(i) $2u_{xx} + u_{xy} - u_{yy} + u_x - 3u = 0$;
(ii) $9u_{xx} + 12u_{xy} + 4u_{yy} - 2u_y = 0$. [4 marks]

(c) Find the general solution $u = u(x, y)$ of the PDE $u_{xy} = 8xy$. [3 marks]

2. Using the method of characteristics and clearly explaining the steps in your solutions, solve the following boundary value problems:

(a) $\begin{cases} u_x - 4u_y = 0, \\ u(0, y) = \cos(y); \end{cases}$ (b) $\begin{cases} u_x - 3x^2 u_y = 0, \\ u(0, y) = y^2. \end{cases}$

[7,9 marks]

3. Suppose the temperature, u, of a thin rod satisfies the heat equation $u_t = u_{xx}$ on the domain

$$R := \{(x, t) \in \mathbb{R}^2 : x \in [0, 4], t \in [0, 5]\},$$

and the following initial and boundary conditions:

$$\begin{cases} u(x, 0) = 4\sin(5\pi x/24), & 0 \le x \le 4; \\ u(0, t) = 0, & 0 < t \le 5; \\ u(2, t) = 2e^{-t}, & 0 < t \le 5. \end{cases}$$

(a) Specifically for the domain R defined above, write down the definition of the *parabolic boundary*, Π. [3 marks]

(b) State the maximum and minimum principles for the heat equation. [2 marks]

(c) Without solving the equation and stating clearly any results you use, find the minimum and maximum temperatures of the rod for all $(x, t) \in R$. [8 marks]

(d) State the values of x and t for which the temperature is maximised. [2 marks]

4. Consider the Cauchy problem for the homogeneous wave equation on an infinite string:

$$\begin{cases} u_{tt} - c^2 u_{xx} = h(x,t) & \text{for } x \in \mathbb{R},\ t > 0, \\ u(x,0) = f(x) & x \in \mathbb{R}, \\ u_t(x,0) = g(x) & x \in \mathbb{R}. \end{cases} \tag{1}$$

(a) Which physical quantity does the constant c describe? [1 mark]

(b) Suppose g and h are both the zero function and $f \in C^2(\mathbb{R})$ is such that $f(x)$ is non-zero for all $x \in (0,1)$ and is zero otherwise. Stating clearly any facts about wave equation solutions you use, give in terms of c both the smallest and largest positive values of t for which $u(5,t)$ is non-zero. [3 marks]

(c) Stating clearly and precisely any results you use, find the solution of problem (1) for all $x \in \mathbb{R}$, $t > 0$, with $c = 1$, and f, g, and h given by

$$f(x) = x^2; \qquad g(x) = \cos(3x); \qquad h(x,t) = xe^{-t}.$$

[14 marks]

5. By seeking a separable solution to the following Laplace equation problem on the unit square:

$$\begin{cases} u_{xx} + u_{yy} = 0, & x \in [0,1],\ y \in [0,1], \\ u(x,0) = u(x,1) = 0, & x \in [0,1], \\ u(0,y) = f(y), & y \in [0,1], \\ u(1,y) = 0, & y \in [0,1], \end{cases}$$

it is possible to show that the solution is of the form

$$u(x,y) = \sum_{n=1}^{\infty} A_n \sinh\left(n\pi(x-1)\right) \sin\left(n\pi y\right).$$

If $f(y) = e^{-3\pi}(1 - e^{6\pi}) \sin(3\pi y)$, find the constants A_n for all $n \in \mathbb{N}$ and hence state the particular solution. [6 marks]

Section B

6. Find the solution of the PDE

$$3\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} - \frac{3}{x}u = 0,$$

which satisfies the condition $u(x, 0) = xe^x$. Simplify your answer as far as possible. [12 marks]

7. The purpose of this question is to show that the maximum principle is not true for the equation

$$u_t - xu_{xx} = 0,$$

which has a variable coefficient.

(a) Verify that $u(x, t) = -2xt - x^2$ is a solution. Find the location of its maximum in the closed rectangle $\{(x, t) \in \mathbb{R}^2 : x \in [-2, 2], t \in [0, 1]\}$. [3,7 marks]

(b) Where exactly does the proof of the maximum principle seen in lectures break down for this equation? [6 marks]

8. The Fourier transform of a function $a \in L^1(\mathbb{R})$, denoted $\bar{a}$, is defined as

$$\bar{a}(\xi) = \int_{-\infty}^{\infty} a(x)e^{i\xi x}\mathrm{d}x.$$

(a) Let $f(x) = e^{-x^2}$. Show that

$$\bar{f}(\xi) = \sqrt{\pi}e^{-\xi^2/4}.$$

You may use that $f \in L^1(\mathbb{R})$ *without proof.* [5 marks]

Let u be a solution to the following problem for the Poisson equation (the inhomogeneous version of Laplace's equation) in a strip:

$$\begin{cases} u_{xx} + u_{yy} = e^{-x^2}, & x \in \mathbb{R}, \quad y \in (0, 1); \\ u(x, 0) = 0, & x \in \mathbb{R}; \\ u(x, 1) = 0, & x \in \mathbb{R}. \end{cases} \tag{2}$$

(b) Classify Poisson's equation as hyperbolic, parabolic or elliptic. [2 marks]

(c) Clearly stating any results that you use, show that the Fourier transform (taken with respect to x) of u satisfies

$$\frac{\partial^2 \bar{u}}{\partial y^2} - \xi^2 \bar{u}(\xi, y) = \sqrt{\pi}e^{-\xi^2/4}.$$

[4 marks]

(d) Hence find an explicit expression for $\bar{u}(\xi, y)$, the Fourier transform of the solution to problem (2). [10 marks]

(e) Write an expression for $u(x, y)$, the solution to problem (2), in terms of $\bar{u}(\xi, y)$. You are not required to evaluate the integral. [1 mark]