Solutions 4

2024–25

1. *By considering the energy integral*

$$E(t) = \frac{1}{2}\int_0^l u^2(x,t)\mathrm{d}x,$$

prove that the following heat equation problem has a unique solution.

$$\begin{cases} u_t = c^2 u_{xx}, \quad 0 \le x \le l,\ 0 \le t < T, \\ u(x,0) = f(x), \quad 0 \le x \le l, \\ u(0,t) = \phi(t), \quad u(l,t) = \psi(t), \quad 0 < t < T. \end{cases}$$

Solution: Suppose for a contradiction that there are two distinct solutions, u_1 and u_2 say. Their difference $u = u_1 - u_2$ satisfies the problem

$$\begin{cases} u_t = c^2 u_{xx}, \quad 0 \le x \le l,\ 0 \le t < T, \\ u(x,0) = 0, \quad 0 \le x \le l, \\ u(0,t) = 0, \quad u(l,t) = 0, \quad 0 < t < T. \end{cases}$$

We compute $\frac{\mathrm{d}E(t)}{\mathrm{d}t}$ using integration by parts:

$$\frac{\mathrm{d}E(t)}{\mathrm{d}t} = \frac{1}{2}\int_0^l 2uu_t\mathrm{d}x = c^2\int_0^l uu_{xx}\mathrm{d}x = -c^2\left(\int_0^l u_x^2\mathrm{d}x - u(x,t)u_x(x,t)\Big|_0^l\right).$$

$$= -c^2\int_0^l u_x^2\mathrm{d}x \le 0.$$

Hence E(t) is decreasing. Moreover, we note that $E(t) \ge 0$ for all t, and $E(0) = 0$. Therefore $E(t) = 0$ for all $t \ge 0$, which implies $u = 0$, that is $u_1 = u_2$. This contradicts our original supposition that u_1 and u_2 were distinct solutions and completes the proof.

2. *Using the maximum principle for the heat equation $u_t = c^2 u_{xx}$ on the rectangle $R := [0,l] \times [0,T]$, prove the* minimum principle*: the solution to the heat equation on R attains its minimum on the parabolic boundary* $\Pi = ([0,l] \times \{0\}) \cup (\{0\} \times [0,T]) \cup (\{l\} \times [0,T])$.

NB: You do not need to re-prove the maximum principle.

Solution: Let $v = -u$. Then v satisfies the heat equation (since $v_t - c^2 v_{xx} = -u_t + c^2 u_{xx} = 0$) and so obeys the maximum principle; that is, v attains its maximal value on the parabolic boundary Π. We deduce that u attains its minimum value on Π.

3. *Suppose u satisfies the heat equation $u_t = u_{xx}$ on the domain $R = [0,1] \times [0,100]$ with the following initial and boundary conditions*

$$\begin{cases} u(x,0) = 0, \quad 0 \le x \le 1, \\ u(0,t) = te^{-t}, \quad u(1,t) = 0, \ t \ge 0. \end{cases}$$

Find constants m, M such that $m \le u(x,t) \le M$ for all $(x,t) \in R$.

Solution: Since we know that u satisfies the heat equation, it must attain its maximal value along the parabolic boundary. Similarly the minimal value is obtained on the parabolic boundary (see previous question). It is therefore enough to calculate the maximum of te^{-t} which is e^{-1} attained at $t = 1$. Thus $0 \le u(x,t) \le e^{-1}$.

4. Compute the Fourier transform of $e^{-a|x|}$, where $a > 0$ is a constant.

Solution

$$\int_{-\infty}^{\infty} e^{-a|x|} e^{i\xi x} \mathrm{d}x = \int_{-\infty}^{\infty} e^{-a|x|+i\xi x} \mathrm{d}x = \int_{-\infty}^{0} e^{(a+i\xi)x} \mathrm{d}x + \int_{0}^{\infty} e^{(i\xi-a)x} \mathrm{d}x$$

$$= \left[\frac{e^{(a+i\xi)x}}{a+i\xi} \right]_{-\infty}^{0} + \left[\frac{e^{(i\xi-a)x}}{i\xi-a} \right]_{0}^{\infty}$$

$$= \frac{1}{a+i\xi} - \frac{1}{i\xi-a} = \frac{2a}{a^2+\xi^2}$$

5. A function $f \in L^1(\mathbb{R})$ has Fourier transform given by $\bar{f}(\xi) = e^{-\xi^2/8}$.

   (a) What is the Fourier transform of f' (i.e. the transform of the derivative of f)?

   (b) Use the Fourier inversion theorem to find $f(x)$.

   (c) Using your answers to (a) and (b), along with any theorems or properties relating to the Fourier transform that you know, deduce the function g whose Fourier transform is given by

   $$\bar{g}(\xi) = 2\xi e^{-\xi^2/4}.$$

   Give your final answer explicitly in closed form (i. e. not as an integral).

Solution

(a) By the derivative theorem, $\mathcal{F}\{f'\} = -i\xi \mathcal{F}\{f\} = -i\xi e^{-\xi^2/8}$.

(b) Noting that $\bar{f} \in L^1(\mathbb{R})$, we have from the Fourier inversion theorem that

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{f}(\xi) e^{-i\xi x} \mathrm{d}\xi = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\xi^2/8} e^{-i\xi x} \mathrm{d}\xi = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\frac{1}{8}(\xi^2+8i\xi x)} \mathrm{d}\xi$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\frac{1}{8}(\xi+4ix)^2} e^{-2x^2} \mathrm{d}\xi = \frac{e^{-2x^2}}{2\pi} \int_{-\infty}^{\infty} e^{-\frac{1}{8}(\xi+4ix)^2} \mathrm{d}\xi = \frac{e^{-2x^2}}{2\pi} \int_{-\infty}^{\infty} e^{-\frac{y^2}{8}} \mathrm{d}y$$

$$= \frac{2\sqrt{2} e^{-2x^2}}{2\pi} \int_{-\infty}^{\infty} e^{-z^2} \mathrm{d}z = \sqrt{\frac{2}{\pi}} e^{-2x^2}, \text{ since } \int_{-\infty}^{\infty} e^{-z^2} \mathrm{d}z = \sqrt{\pi}.$$

Here we have used the substitutions $y = \xi + 4ix$ and $z = \frac{\sqrt{2}y}{4}$.

(c) Note that $\bar{g}(\xi) = 2\xi e^{-\xi^2/4} = 2i(e^{-\xi^2/8})(-i\xi e^{-\xi^2/8}) = 2i\mathcal{F}\{f\}\mathcal{F}\{f'\} = 2i\mathcal{F}\{f * f'\}$ by the convolution theorem. Fix x. Then:

$$\begin{aligned}
g(x) = 2i(f * f')(x) = 2i \int_{-\infty}^{\infty} f(x-y)f'(y)\mathrm{d}y \text{ (by definition of convolution)} \\
&= 2i \int_{-\infty}^{\infty} \left(\sqrt{\frac{2}{\pi}} e^{-2(x-y)^2}\right)\left(-4y\sqrt{\frac{2}{\pi}} e^{-2y^2}\right) \mathrm{d}y \\
&= -\frac{16i}{\pi} \int_{-\infty}^{\infty} y e^{-2x^2+4xy-4y^2} \mathrm{d}y = -\frac{16ie^{-2x^2}}{\pi} \int_{-\infty}^{\infty} y e^{-4(y^2-xy)} \mathrm{d}y \\
&= -\frac{16ie^{-x^2}}{\pi} \int_{-\infty}^{\infty} y e^{-4(y-x/2)^2} \mathrm{d}y = -\frac{16ie^{-x^2}}{\pi} \int_{-\infty}^{\infty} \left(z + \frac{x}{2}\right) e^{-4z^2} \mathrm{d}z \\
&= -\frac{16ie^{-x^2}}{\pi} \left\{ \int_{-\infty}^{\infty} z e^{-4z^2} \mathrm{d}z + \frac{x}{2} \int_{-\infty}^{\infty} e^{-4z^2} \mathrm{d}z \right\} \\
&= -\frac{16ixe^{-x^2}}{2\pi} \int_{-\infty}^{\infty} e^{-4z^2} \mathrm{d}z = -\frac{16ixe^{-x^2}}{2\pi} \int_{-\infty}^{\infty} e^{-\alpha^2} \frac{\mathrm{d}\alpha}{2} = -\frac{4i}{\sqrt{\pi}} x e^{-x^2}.
\end{aligned}$$

6. (a) Let $f \in L^1(\mathbb{R})$ be a real valued function. Show that its Fourier transform $\bar{f}$ satisfies

$$\bar{f}^*(\xi) = \bar{f}(-\xi),$$

where $\bar{f}^*$ denotes the complex conjugate of $\bar{f}$ (this property is called Hermitian symmetry).

(b) Derive a similar relationship between the transform of a purely imaginary function g and its complex conjugate.

(c) Show that the Fourier transform of a real even function is real.

(d) Show that the Fourier transform of a real odd function is imaginary.

(e) Show that the Fourier transform of an even function is even.

Solution

(a) Note that

$$\begin{aligned}
\bar{f}^*(\xi) = \left(\int_{-\infty}^{\infty} f(x) e^{i\xi x} \mathrm{d}x\right)^* &= \left(\int_{-\infty}^{\infty} f(x)[\cos(\xi x) + i\sin(\xi x)]\mathrm{d}x\right)^* \\
= \int_{-\infty}^{\infty} f(x)[\cos(\xi x) - i\sin(\xi x)]\mathrm{d}x &= \int_{-\infty}^{\infty} f(x) e^{-i\xi x} \mathrm{d}x. = \bar{f}(-\xi)
\end{aligned}$$

(b) Note that for a real function h defined by $g(x) = ih(x)$,

$$\bar{g}^*(\xi) = \left(\int_{-\infty}^{\infty} g(x)e^{i\xi x}\mathrm{d}x\right)^* = \left(\int_{-\infty}^{\infty} ih(x)[\cos(\xi x) + i\sin(\xi x)]\mathrm{d}x\right)^*$$

$$= \int_{-\infty}^{\infty} -ih(x)\cos(\xi x) - h(x)\sin(\xi x)]\mathrm{d}x = \int_{-\infty}^{\infty} -g(x)[\cos(\xi x) - i\sin(\xi x)]\mathrm{d}x$$

$$= \int_{-\infty}^{\infty} -g(x)e^{-i\xi x}\mathrm{d}x = -\bar{g}(-\xi).$$

(c) Let f be real and even. Then

$$\bar{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{i\xi x}\mathrm{d}x = \int_{-\infty}^{\infty} f(x)[\cos(\xi x) + i\sin(\xi x)]\mathrm{d}x = \int_{-\infty}^{\infty} f(x)\cos(\xi x)\mathrm{d}x,$$

which is clearly real.

(d) Let f be real and odd. Then

$$\bar{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{i\xi x}\mathrm{d}x = \int_{-\infty}^{\infty} f(x)[\cos(\xi x) + i\sin(\xi x)]\mathrm{d}x = \int_{-\infty}^{\infty} if(x)\sin(\xi x)\mathrm{d}x,$$

which is clearly imaginary.

(e) Let f be even. Then

$$\bar{f}(-\xi) = \int_{-\infty}^{\infty} f(x)e^{-i\xi x}\mathrm{d}x = \int_{-\infty}^{\infty} f(x)e^{i\xi(-x)}\mathrm{d}x.$$

Make the substitution $y = -x$:

$$\int_{-\infty}^{\infty} f(x)e^{i\xi(-x)}\mathrm{d}x = -\int_{\infty}^{-\infty} f(-y)e^{i\xi y}\mathrm{d}y = \int_{-\infty}^{\infty} f(-y)e^{i\xi y}\mathrm{d}y,$$

and since f is even, we have that

$$\int_{-\infty}^{\infty} f(-y)e^{i\xi y}\mathrm{d}y = \int_{-\infty}^{\infty} f(y)e^{i\xi y}\mathrm{d}y = \bar{f}(\xi).$$

That is, the Fourier transform of an even function is even.

7. Find the Fourier transform with respect to x of the solution to the following boundary value problem for Laplace's equation:

$$\begin{cases} u_{xx} + u_{yy} = 0, & x \in \mathbb{R},\ y \in [0, 1]; \\ u(x, 0) = 0, & x \in \mathbb{R}; \\ u(x, 1) = e^{-|x|}, & x \in \mathbb{R}. \end{cases}$$

Solution: Define $\bar{u}(\xi, y)$ as the Fourier transform of u with respect to x. That is:

$$\mathcal{F}\{u\} = \bar{u}(\xi, y) = \int_{-\infty}^{\infty} u(x, y)e^{i\xi x}\mathrm{d}x.$$

Then the derivative theorem gives that

$$\mathcal{F}\{u_{xx}\} = (-i\xi)(-i\xi)\mathcal{F}\{u\} = -\xi^2\bar{u},$$

while

$$\mathcal{F}\{u_{yy}\} = \frac{\partial^2\bar{u}}{\partial y^2}$$

(since FTs are taken wrt x, not y here). Thus applying Fourier transforms to the PDE (Laplace's equation) yields

$$\frac{\partial^2\bar{u}}{\partial y^2} - \xi^2\bar{u} = 0.$$

Since this only has differentiation wrt one independent variable, we can solve in a similar fashion to an ODE. The auxiliary equation is $m^2 - \xi^2 = 0$, whence

$$\bar{u}(\xi, y) = A(\xi)e^{|\xi|y} + B(\xi)e^{-|\xi|y}.$$

We will now apply the boundary conditions to find the functions A and B. Taking FTs wrt x of the BCs gives

$$\begin{cases} u(\xi, 0) = 0, & \xi \in \mathbb{R}; \\ u(\xi, 1) = \frac{2}{1+\xi^2}, & \xi \in \mathbb{R} \text{ (see Q1 with } a = 1). \end{cases}$$

The first condition yields $A(\xi) = -B(\xi)$. The second condition then gives

$$\frac{2}{1+\xi^2} = A(\xi)\left(e^{|\xi|} - e^{-|\xi|}\right) = 2A(\xi)\sinh(|\xi|).$$

Thus $A(\xi) = \frac{\operatorname{csch}(|\xi|)}{1+\xi^2}$ and so $B(\xi) = -\frac{\operatorname{csch}(|\xi|)}{1+\xi^2}$. Finally, therefore, the Fourier transform wrt x of u is given by

$$\bar{u}(\xi, y) = \frac{\operatorname{csch}(|\xi|)}{1+\xi^2}\left(e^{|\xi|y} - e^{-|\xi|y}\right) = \frac{2\operatorname{csch}(|\xi|)\sinh(|\xi|y)}{1+\xi^2}.$$

8. (Harder) Prove that the function

$$f(x) = \frac{2\sin(\xi/2)}{\xi}$$

does not belong to $L^1(\mathbb{R})$.

Solution: Let $f(\xi) = \frac{2\sin(\xi/2)}{\xi}$. Noting that the integrand is non-negative, it is enough to show that $\int_0^\infty |f(\xi)|\mathrm{d}\xi$ does not converge. Consider the intervals

$$I_k = [2k\pi, 2(k+1)\pi], \quad k = 0, \ldots, n,$$

and define f_k to be the restriction of f to the interval I_k. Then on each interval I_k, f_k is a continuous, integrable function satisfying $|f(\xi)| \geq \frac{|\sin \frac{\xi}{2}|}{(k+1)\pi}$. Thus for $n \in \mathbb{N}$,

$$\int_0^{(n+1)\pi} |f(\xi)| \mathrm{d}\xi = \sum_{k=0}^{n} \int_{2k\pi}^{2(k+1)\pi} \frac{2|\sin(\xi/2)|}{\xi} \mathrm{d}\xi \geq \sum_{k=0}^{n} \int_{2k\pi}^{2(k+1)\pi} \frac{|\sin(\xi/2)|}{(k+1)\pi} \mathrm{d}\xi = \sum_{k=0}^{n} \frac{4}{(k+1)\pi} = \frac{4}{\pi} \sum_{m=1}^{n+1} \frac{1}{m}.$$

Letting $n \to \infty$ and recalling that the series $\sum_{m=1}^{\infty} \frac{1}{m}$ is divergent, we deduce that $f \notin L^1(\mathbb{R})$.