Solutions 3

2024–25

1. Using the reflection method and the d'Alembert formula, solve the Cauchy problem for the wave equation for a half-infinite string

$$\begin{cases} u_{tt} - 9u_{xx} = 0, \quad x > 0,\ t > 0, \\ u(x,0) = x^2 e^{-x}, \quad u_t(x,0) = xe^{-x}, \\ u(0,t) = 0 \quad \text{(Dirichlet boundary condition)}. \end{cases}$$

Solution. We apply the principle of odd reflection. Define $\tilde{f}$, $\tilde{g}$ and $\tilde{u}$ by

$$\tilde{f}(x) = \begin{cases} x^2 e^{-x}, & x \geq 0 \\ -x^2 e^{x}. & x < 0 \end{cases}$$

$$\tilde{g}(x) = \begin{cases} xe^{-x}, & x \geq 0 \\ xe^{x}. & x < 0 \end{cases}$$

$$\tilde{u}(x) = \begin{cases} u(x), & x \geq 0 \\ -u(-x). & x < 0 \end{cases}$$

Then $\tilde{u}$ satisfies the Cauchy problem defined for the whole line

$$\begin{cases} \tilde{u}_{tt} - 9\tilde{u}_{xx} = 0, \quad x \in \mathbb{R},\ t > 0, \\ \tilde{u}(x,0) = \tilde{f}(x), \quad \tilde{u}_t(x,0) = \tilde{g}(x), \end{cases}$$

and because of the odd reflection satisfies $u(0,t) = 0$ for all $t > 0$. This has solution by d'Alembert's formula:

$$u(x,y) = \frac{1}{2}\{\tilde{f}(x+3t) + \tilde{f}(x-3t)\} + \frac{1}{6}\int_{x-3t}^{x+3t} \tilde{g}(\lambda)\mathrm{d}\lambda.$$

Considering the cases $0 < x < 3t$ and $x > 3t$ separately as in the lectures, the solution is given by

$$u(x,y) = \begin{cases} \frac{1}{2}\left\{(x+3t)^2 e^{-x-3t} + (x-3t)^2 e^{-x+3t}\right\} + \frac{1}{6}\int_{x-3t}^{x+3t} \lambda e^{-\lambda}\mathrm{d}\lambda, & x > 3t \\ \frac{1}{2}\left\{(x+3t)^2 e^{-x-3t} - (x-3t)^2 e^{x-3t}\right\} + \frac{1}{6}\int_{3t-x}^{x+3t} \lambda e^{-\lambda}\mathrm{d}\lambda, & 0 < x < 3t, \end{cases}$$

which evaluates as

$$u(x,y) = \begin{cases} \frac{1}{2}\left\{(x+3t)^2 e^{-x-3t} + (x-3t)^2 e^{3t-x}\right\} - \frac{1}{6}\left((1+x+3t)e^{-x-3t} - (1+x-3t)e^{-x+3t}\right), & x > 3t \\ \frac{1}{2}\left\{(x+3t)^2 e^{-x-3t} - (x-3t)^2 e^{x-3t}\right\} - \frac{1}{6}\left((1+x+3t)e^{-x-3t} - (1+3t-x)e^{x-3t}\right), & 0 < x < 3t. \end{cases}$$

2. Find the solution of the inhomogenous wave equation

$$\begin{cases} u_{tt} - c^2 u_{xx} = x\cos t, & x \in \mathbb{R}, t > 0, \\ u(x,0) = 0, \quad u_t(x,0) = 0. \end{cases}$$

Solution. The Duhamel principle yields that the solution is given by

$$u(x,t) = \frac{1}{2c} \int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} y \cos\tau \mathrm{d}y \mathrm{d}\tau = x(1-\cos t).$$

3. Let $l > 0$ and $m, n \in \mathbb{N}$. Prove that

$$\int_0^l \sin\frac{m\pi x}{l} \sin\frac{n\pi x}{l} \mathrm{d}x = \begin{cases} \frac{l}{2}, & n = m, \\ 0, & n \neq m. \end{cases}$$

Solution: We use the trigonometric identity

$$\sin\alpha \sin\beta = \frac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right],$$

giving

$$\int_0^l \sin\left(\frac{n\pi x}{l}\right) \sin\left(\frac{m\pi x}{l}\right) \mathrm{d}x = \frac{1}{2}\int_0^l \cos\left(\frac{(n-m)\pi x}{l}\right) \mathrm{d}x - \frac{1}{2}\int_0^l \cos\left(\frac{(n+m)\pi x}{l}\right) \mathrm{d}x.$$

First

$$\begin{aligned} \int_0^l \cos\left(\frac{(n+m)\pi x}{l}\right) \mathrm{d}x &= \frac{l}{(n+m)\pi} \sin\left(\frac{(n+m)\pi x}{l}\right) |_{x=0}^{x=l} \\ &= \frac{l}{(n+m)\pi}[\sin((n+m)\pi) - \sin(0)] = 0. \end{aligned}$$

Similarly, for $n \neq m$,

$$\begin{aligned} \int_0^l \cos\left(\frac{(n-m)\pi x}{l}\right) \mathrm{d}x &= \frac{l}{(n-m)\pi} \sin\left(\frac{(n-m)\pi x}{l}\right) |_{x=0}^{x=l} \\ &= \frac{l}{(n-m)\pi}[\sin((n-m)\pi) - \sin(0)] = 0. \end{aligned}$$

But, for $n = m$,

$$\int_0^l \cos\left(\frac{(n-m)\pi x}{l}\right) \mathrm{d}x = \int_0^l \cos(0)\mathrm{d}x = \int_0^l \mathrm{d}x = l.$$

Therefore, for $n \neq m$, we have

$$\int_0^l \sin\frac{n\pi x}{l} \sin\frac{m\pi x}{l} \mathrm{d}x = 0,$$

while for $n = m$, we have

$$\int_0^l \sin\frac{n\pi x}{l} \sin\frac{m\pi x}{l} \mathrm{d}x = \frac{l}{2}.$$

4. Using the method of separation of variables, solve the initial boundary value problem

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, \quad x \in [0, \pi],\ t > 0, \\ u(x,0) = \sin 3x, \quad u_t(x,0) = 0, \text{ initial conditions.} \\ u(0,t) = 0 \quad u(l,t) = 0,\ t > 0 \text{ boundary conditions.} \end{cases}$$

(You may use formulae derived in the lectures if you wish to prevent your solutions becoming too long, but make sure you understand their derivation.)

Solution: The solution is given by

$$u(x,t) = \sum_{k=1}^{\infty} \left(A_k \cos \frac{k\pi ct}{l} + B_k \sin \frac{k\pi ct}{l} \right) \sin \frac{k\pi x}{l},$$

(see lecture notes for derivation); in this case $l = \pi$ and so this reduces to

$$u(x,t) = \sum_{k=1}^{\infty} (A_k \cos(kct) + B_k \sin(kct)) \sin(kx),$$

The condition $u_t(x,0) = 0$ implies that $B_k = 0$ for all k. The condition $u(x,0) = \sin(3x)$ implies that $A_3 = 1$, and $A_k = 0$ for all other $k \neq 3$. Therefore the required solution is

$$u(x,t) = \cos(3ct) \sin(3x).$$

5. Using the method of separation of variables, find the general solution to the initial boundary value problem for the one dimensional heat equation with endpoints held at zero temperature.

$$\begin{cases} u_t - c^2 u_{xx} = 0, \quad x \in [0, l],\ t > 0, \\ u(x,0) = f(x), \\ u(0,t) = 0, \quad u(l,t) = 0,\ t > 0. \end{cases}$$

Solution: Assume a separable solution, i.e. seek u in the form $u(x,t) = X(x)T(t)$. Substitution of this form into the heat equation yields

$$(X(x)T(t))_t - c^2(X(x)T(t))_{xx} = 0 \Rightarrow X(x)T'(t) - c^2 T(t) X''(x) = 0.$$

Rearranging yields

$$\frac{T'(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda, \text{ say,}$$

where $-\lambda$ is a constant since the two sides of the preceeding equation are functions solely of the independent variables t and x respectively. The boundary conditions imply that $X(0) = 0$ and $X(l) = 0$. Consequently we can consider the problem for $X(x)$:

$$\begin{cases} X''(x) + \lambda X(x) = 0; \\ X(0) = X(l) = 0. \end{cases}$$

This is exactly the same problem as we met in the lectures and indeed in Q4. It has infinitely many solutions (eigenfunctions), each of the form $X_k(x) = \sin(k\pi x/l)$, with $k \in \mathbb{N}$. The eigenvalues (derived as in the lecture) are $\lambda_k = (k\pi/l)^2$ for $k \in \mathbb{N}$.

The problem for $T(t)$ (sought only in the case $\lambda = \lambda_k$ as we seek non-trivial solutions) becomes

$$T_k'(t) + c^2 \lambda_k T(t) = 0, \qquad k \in \mathbb{N}.$$

This first order linear ODE has solutions (solved by integrating factor method)

$$T_k(t) = a_k \exp\left(-c^2 \lambda_k t\right),$$

where a_k ($k \in \mathbb{N}$) are arbitrary constants. Since the heat equation is linear homogeneous, the superposition principle gives that any linear combination of solutions is itself a solution, so

$$u(x,t) = \sum_{k=1}^{\infty} u_k(x,t) = \sum_{k=1}^{\infty} X_k(x) T_k(t) = \sum_{k=1}^{\infty} a_k \sin\left(\frac{k\pi x}{l}\right) \exp\left(-\frac{c^2 k^2 \pi^2 t}{l^2}\right).$$

Finally we must find the constants a_k such that $u(x,0) = f(x)$. That is, find a_k such that

$$\sum_{k=1}^{\infty} a_k \sin(k\pi x/l) = f(x); \tag{1}$$

this is a Fourier sine series problem. We solve by fixing $m \in \mathbb{N}$, multiplying both sides of (1) by $\sin(m\pi x/l)$ and integrating w.r.t. x term by term between 0 and l. The result from Q3 gives

$$a_k = \frac{2}{l} \int_0^l f(x) \sin\left(\frac{k\pi x}{l}\right) \mathrm{d}x.$$

6. Using the method of separation of variables, find the particular solution to the following Laplace equation initial boundary value problem:

$$\begin{cases} u_{xx} + u_{yy} = 0, & x \in [0,1],\ y \in [0,1]; \\ u(0,y) = u(1,y) = 0, & y \in [0,1]; \\ u(x,1) = 0, & x \in [0,1]; \\ u(x,0) = 4\sin(5\pi x), & x \in [0,1]. \end{cases}$$

Solution: Assume a separable solution, i.e. seek u in the form $u(x,t) = X(x)Y(y)$. Substitution of this form into the heat equation yields

$$(X(x)Y(y))_{xx} + (X(x)Y(y))_{yy} = 0 \Rightarrow X''(x)Y(y) + X(x)Y''(y) = 0.$$

Rearranging yields

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = -\lambda, \text{ say,}$$

where $-\lambda$ is a constant since the two sides of the preceeding equation are functions solely of the independent variables x and y respectively. The boundary conditions imply that $X(0) = 0$ and $X(1) = 0$. Consequently we can consider the problem for $X(x)$:

$$\begin{cases} X''(x) + \lambda X(x) = 0; \\ X(0) = X(1) = 0. \end{cases}$$

To obtain (infinitely many) non-trivial solutions (indexed by $n \in \mathbb{N}$), let $\lambda_n = \omega_n^2$. Solving the second order ODE gives

$$X_n(x) = A_n \sin(\omega_n x) + B_n \cos(\omega_n x).$$

The condition $X_n(0) = 0$ implies that $B_n = 0 \forall n \in \mathbb{N}$. For non-trivial solutions that satisfy $X_n(1) = 0$, $\omega_n = n\pi$, so

$$X_n(x) = B_n \sin(n\pi x), \qquad n \in \mathbb{N}.$$

The problem for y becomes $Y_n''(y) = n^2\pi^2 Y(y)$. Solving these second order linear constant coefficient ODEs gives

$$Y_n(y) = C_n e^{n\pi y} + D_n e^{-n\pi y} = E_n \cosh(n\pi(y-1)) + F_n \sinh(n\pi(y-1))$$

(Here the arbitrary constants E_n and F_n are related to C_n and D_n through $C_n = e^{-n\pi}(E_n + F_n)$ and $D_n = e^{n\pi}(E_n - F_n)$). The condition $u(x,1) = 0$ implies $E_n = 0 \forall n \in \mathbb{N}$.

The superposition principle yields

$$u(x,y) = \sum_{n=1}^{\infty} G_n \sin(n\pi x) \sinh(n\pi(y-1)).$$

Finally the condition $u(x,0) = 4\sin(5\pi x)$ gives that

$$-\sum_{n=1}^{\infty} G_n \sin(n\pi x) \sinh(n\pi) = 4\sin(5\pi x),$$

whence $G_5 = -4\text{cosech}(5\pi)$ and $G_n = 0 \forall n \in \mathbb{N}\backslash\{5\}$. The particular solution is therefore

$$u(x,y) = -4\text{cosech}(5\pi) \sin(5\pi x) \sinh(5\pi(y-1)).$$