Solutions 1

2024–25

1. (a) $\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{2y}{x} = 3x^3$: The ODE is linear and first order. We multiply throughout by an integrating factor $\exp(-\int \frac{2}{x}\mathrm{d}x) = \exp(-2\ln x) = x^{-2}$, giving

$$x^{-2}\frac{\mathrm{d}y}{\mathrm{d}x} - 2x^{-3}y = 3x \Leftrightarrow \frac{\mathrm{d}}{\mathrm{d}x}\left\{x^{-2}y\right\} = 3x \Rightarrow y = \frac{3x^4}{2} + cx^2,$$

where c is an arbitrary constant.

(b) $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2\frac{\mathrm{d}y}{\mathrm{d}x} - 3y = 0$: The ODE is second order, linear, with constant coefficients. The auxiliary equation is $m^2 - 2m - 3 = 0 \Leftrightarrow (m-3)(m+1) = 0$ and so the roots of the auxiliary equation are $m = -1$ and $m = 3$. Consequently the general solution is $y = Ae^{-x} + Be^{3x}$, where A and B are arbitrary constants.

(c) $y\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3}{x^2y}$: The equation is separable, so

$$\int y^2\mathrm{d}y = 3\int x^{-2}\mathrm{d}y + c \Rightarrow \frac{y^3}{3} = c - 3x^{-1} \Rightarrow y = \left(C - \frac{9}{x}\right)^{1/3},$$

where c, C are arbitrary constants.

(d) $\frac{\mathrm{d}y}{\mathrm{d}x} - y = 0$: $y = Ae^x$ (A arbitrary, solved either by inspection, integrating factor of e^{-x} or separable variables techniques).

(e) $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + y = 0$: The ODE is second order, linear, with constant coefficients. The auxiliary equation is $m^2 + 1 = 0$ and so the roots of the auxiliary equation are $m = \pm i$. It follows that $y = A\cos x + B\sin x$, where A, B are arbitrary constants.

2. (a) $u_x + xu_y = \sin x$: First order. We write this in the form $Lu = f(x,y)$, where $L = \frac{\partial}{\partial x} + x\frac{\partial}{\partial x}$ and $f(x,y) = \sin x$. For any C^1 functions u and v, it follows from the linearity of partial differentiation that $L(u+v) = Lu + Lv$. Moreover, for a constant c, $L(cu) = cLu$. L is therefore a linear operator, and so the PDE is linear inhomogeneous.

(b) $u_{xx} + uu_x = 0$: Second order. We write this in the form $Lu = 0$, where $L = \frac{\partial^2}{\partial x^2} + (\cdot)\frac{\partial}{\partial x}$. This is a non-linear operator due to the term $(\cdot)\frac{\partial}{\partial x}$. This can be seen by noting that for a constant c,

$$L(cu) = \frac{\partial^2(cu)}{\partial x^2} + cu\frac{\partial}{\partial x}(cu) = c\frac{\partial^2 u}{\partial x^2} + c^2u\frac{\partial u}{\partial x} \neq c\left(\frac{\partial^2 u}{\partial x^2} + u\frac{\partial u}{\partial x}\right) = cLu.$$

It follows that the PDE is non-linear.

(c) $u_x + uu_y = u$: First order. The PDE is non-linear due to the presence of the term uu_y due to the same argument as in part (b).

(d) $u_x + 2u_y + 3 = x$: First order. We can rearrange this PDE (leaving all terms in the dependent variable u on the left hand side and other terms on the right hand side) to give $u_x + 2u_y = x - 3$. We can write this in the form $Lu = f(x, y)$, where $L = \frac{\partial}{\partial x} + 2\frac{\partial}{\partial y}$ and $f(x, y) = x - 3$. By linearity of partial differentiation, we note that L is a linear operator. It follows that the PDE is linear inhomogeneous. (**NB:** the operator $\frac{\partial}{\partial x} + 2\frac{\partial}{\partial y} + 3$ is non-linear, so the initial rearranging was important.)

3. Find the general solution $u = u(x, y)$ of the PDE $u_{xy} = 3xy$.

   Integrating with respect to y gives $u_x = \frac{3xy^2}{2} + h(x)$. Integrating this with respect to x gives $u = \frac{3x^2y^2}{4} + \int h(x)\mathrm{d}x + g(y)$, and since integrating an arbitrary function of x yields another arbitrary function of x, we can write the general solution as $u = \frac{3x^2y^2}{4} + f(x) + g(y)$.

4. (a) Find the general solution $u = u(x, t)$ of the PDE $4u_x + 3u_t = 0$. Hence find the solution of the PDE $4u_x + 3u_t = 0$ satisfying the initial condition (for $t = 0$): $u(x, 0) = \cos x$.

   We apply the method of characteristics. The PDE can be written in the form $(4, 3) \cdot \nabla u = 0$ and so u is constant along characteristic curves defined by the ODE $\frac{dx}{dt} = \frac{4}{3}$, which has solution $x = \frac{4}{3}t + c$ where $c \in \mathbb{R}$ is constant (one constant for each characteristic curve). It follows that the general solution is $u(x, t) = f(c) = f(x - \frac{4}{3}t)$, where $f \in C^1(\mathbb{R})$ is arbitrary.

   (b) We now apply the initial condition: $u(x, 0) = f(x) = \cos x$. The solution of the PDE is therefore $u(x, t) = \cos(x - \frac{4}{3}t)$.

5. Solve the PDE $x^2yu_x + 3u_y = 0$, with $u(x, 0) = \frac{1}{x}$.

   We can write this in the form $(x^2y, 3) \cdot \nabla u = 0$. That is, solutions are constant along curves whose tangent lies in the direction $(x^2y, 3)$. Such curves are described by the ODE $\frac{\mathrm{d}y}{\mathrm{d}x} = 3/(x^2y)$, which is separable. Solutions of the ODE are of the form $y^2 = c - \frac{6}{x}$. Since u is constant along each characteristic curve and different characteristic curves have different values of c, it follows that $u = f(c) = f(y^2 + 6/x)$. The boundary condition implies $u(x, 0) = 1/x = f(6/x)$, so $f(z) = z/6$. The particular solution is therefore $y = y^2/6 + 1/x$.

6. Solve the linear equation $(1 + x^2)u_x + u_y = 0$.

   We can write this in the form $(1 + x^2, 1) \cdot \nabla u = 0$. Characteristic curves for this equation satisfy $dy/dx = 1/(1+x^2)$. Solving this ODE yields $y = \arctan x + C$. Solutions of the PDE are constant on each characteristic curve. Thus, the general solution of PDE is $u = f(y - \arctan x)$.

7. Solve the equation $(\sqrt{1 - x^2})u_x + u_y = 0$ with the condition $u(0, y) = y$.

   Similarly, the characteristic curves for this equation satisfy $dy/dx = 1/\sqrt{1 - x^2}$, which yields $y = \arcsin x + C$. Thus, solutions take the form $u(x, y) = f(y - \arcsin x)$. The condition $u(0, y) = y$ implies $f(y) = y$, so $u(x, y) = y - \arcsin x$ is the required solution.

8. Using the method of characteristics, solve $u_x + u_y + u = e^{x+2y}$ with $u(x, 0) = 0$.

   Consider the curves defined by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 1, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = 1,$$

   with conditions $x(0) = s$, $y(0) = 0$, implying $x = t + s$ and $y = t$. Along these curves, the PDE reduces to the ODE

$$\frac{\mathrm{d}u}{\mathrm{d}t} + u = e^{3t+s}.$$

Multiply by an integrating factor of e^t to give

$$e^t \frac{\mathrm{d}u}{\mathrm{d}t} + e^t u = e^{4t+s} \Leftrightarrow \frac{\mathrm{d}}{\mathrm{d}t}\left\{e^t u\right\} = e^{4t+s} \Rightarrow u = \frac{1}{4}e^{3t+s} + c(s)e^{-t}.$$

Noting that $t = y$ and $s = x - y$, we transform back to the original problem variables x and y to give

$$u(x, y) = \frac{1}{4}e^{2y+x} + c(x - y)e^{-y}.$$

Applying the condition $u(x, 0) = 0$ yields that $c(x) = -\frac{1}{4}e^x$, and so the required overall solution to the original problem is

$$u(x, y) = \frac{1}{4}\left(e^{x+2y} - e^{x-2y}\right) = \frac{1}{4}e^x\left(e^{2y} - e^{-2y}\right) = \frac{1}{2}e^x \sinh 2y.$$