Assignment 4

2024–25

Starred questions due in at any point before the end of term.

1. *By considering the energy integral

$$E(t) = \frac{1}{2}\int_0^l u^2(x,t)\mathrm{d}x,$$

prove that the following heat equation problem has a unique solution.

$$\begin{cases} u_t = c^2 u_{xx}, \quad 0 \le x \le l,\ 0 \le t < T, \\ u(x,0) = f(x), \quad 0 \le x \le l, \\ u(0,t) = \phi(t), \quad u(l,t) = \psi(t), \quad 0 < t < T. \end{cases}$$

Hint: Suppose for a contradiction that there are two distinct solutions. What problem does their difference satisfy? What sign is $E(t)$? Show that $E(t)$ is a decreasing function and make deductions.

2. Using the maximum principle for the heat equation $u_t = c^2 u_{xx}$ on the rectangle $R := [0,l] \times [0,T]$, prove the *minimum principle*: the solution to the heat equation on R attains its minimum on the parabolic boundary $\Pi = ([0,l]\times\{0\}) \cup (\{0\}\times[0,T]) \cup (\{l\}\times[0,T])$.

NB: You do not need to re-prove the maximum principle.

3. *Suppose u satisfies the heat equation $u_t = u_{xx}$ on the domain $R = [0,1]\times[0,100]$ with the following initial and boundary conditions

$$\begin{cases} u(x,0) = 0, \quad 0 \le x \le 1, \\ u(0,t) = te^{-t}, \quad u(1,t) = 0,\ t \ge 0. \end{cases}$$

Find constants m, M such that $m \le u(x,t) \le M$ for all $(x,t) \in R$.

4. *Compute the Fourier transform of $e^{-a|x|}$, where $a > 0$ is a constant.

5. *A function $f \in L^1(\mathbb{R})$ has Fourier transform given by $\bar{f}(\xi) = e^{-\xi^2/8}$.

   (a) What is the Fourier transform of f' (i.e. the transform of the derivative of f)?

   (b) Use the Fourier inversion theorem to find $f(x)$.

   (c) Using your answers to (a) and (b), along with any theorems or properties relating to the Fourier transform that you know, deduce the function g whose Fourier transform is given by

   $$\bar{g}(\xi) = 2\xi e^{-\xi^2/4}.$$

   Give your final answer explicitly in closed form (i. e. not as an integral).

6. (a) Let $f \in L^1(\mathbb{R})$ be a real valued function. Show that its Fourier transform $\bar{f}$ satisfies

$$\bar{f}^*(\xi) = \bar{f}(-\xi),$$

where $\bar{f}^*$ denotes the complex conjugate of $\bar{f}$ (this property is called Hermitian symmetry).

   (b) Derive a similar relationship between the transform of a purely imaginary function g and its complex conjugate.

   (c) Show that the Fourier transform of a real even function is real.

   (d) Show that the Fourier transform of a real odd function is imaginary.

   (e) Show that the Fourier transform of an even function is even.

7. *Find the Fourier transform with respect to x of the solution to the following boundary value problem for Laplace's equation:

$$\begin{cases} u_{xx} + u_{yy} = 0, & x \in \mathbb{R},\ y \in [0,1]; \\ u(x,0) = 0, & x \in \mathbb{R}; \\ u(x,1) = e^{-|x|}, & x \in \mathbb{R}. \end{cases}$$

8. (Harder) Prove that the function

$$\bar{f}(\xi) = \frac{2\sin(\xi/2)}{\xi}$$

does not belong to $L^1(\mathbb{R})$.