Reed-Muller codes and permutation decoding

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Abstract

We show that the first- and second-order Reed-Muller codes, $R(1,m)$ and $R(2,m)$, can be used for permutation decoding by finding, within the translation group, $(m-1)$- and $(m+1)$-PD-sets for $R(1,m)$ for $m \geq 5,6$, respectively, and $(m-3)$-PD-sets for $R(2,m)$ for $m \geq 8$. We extend the results of Seneviratne [14].

1 Introduction

The first- and second-order Reed-Muller codes, $R(1,m)$ and $R(2,m)$, are binary codes with large minimum weight, being the codes of the affine geometry designs over $\mathbb{F}_2$ of points and $(m-1)$-flats or $(m-2)$-flats, respectively, and with the minimum words the incidence vectors of the blocks. Furthermore, they each have a large automorphism group containing the translation group, making them good candidates for permutation decoding. Seneviratne [14] found 4-PD-sets for the first-order Reed-Muller codes $R(1,m)$ for $m \geq 5$. We extend his method to find $(m-1)$-PD-sets of size $\frac{1}{2}(m^2 + m + 4)$ for $R(1,m)$ for $m \geq 5$, $(m+1)$-PD-sets of size $\frac{1}{6}(m^3 + 5m + 12)$ for $R(1,m)$ for $m \geq 6$, and $(m-3)$-PD-sets of size $\frac{1}{6}(m^3 + 5m + 12)$ for $R(2,m)$ for $m \geq 8$.

We prove the following theorem.

Theorem 1 Let $V = \mathbb{F}_2^n$ and $C_i = \{v \mid v \in V, \text{wt}(v) = i\}$ for $0 \leq i \leq m$. Let $T_u$ denote the translation of $V$ by $u \in V$,

$$A_m = \{T_u \mid u \in C_0 \cup C_1 \cup C_2 \cup C_m\}, \quad B_m = A_m \cup \{T_u \mid u \in C_3\},$$

then

1. $A_m$ is an $(m-1)$-PD-set of size $\frac{1}{2}(m^2 + m + 4)$ for $R(1,m)$ and $m \geq 5$ using the information set $C_0 \cup C_1$;

2. $B_m$ is an $(m+1)$-PD-set of size $\frac{1}{6}(m^3 + 5m + 12)$ for $R(1,m)$ and $m \geq 6$ using the information set $C_0 \cup C_1$;
3. $B_m$ is an $(m - 3)$-PD-set of size $\frac{1}{6}(m^3 + 5m + 12)$ for $R(2, m)$ and $m \geq 8$ using the information set $C_0 \cup C_1 \cup C_2$.

The theorem will follow from Propositions 1, 2 and 3 in Sections 4 and 5. Before stating and proving these propositions, we give some background results and definitions.

2 Background and terminology

Most of the notation will be as in [1], with some exceptions noted. An incidence structure $D = (P, B, I)$, with point set $P$, block set $B$ and incidence $I$ is a $t$-$(v, k, \lambda)$ design, if $|P| = v$, every block $B \in B$ is incident with precisely $k$ points, and every $t$ distinct points are together incident with precisely $\lambda$ blocks. We deal here with the design of points and $t$-flats, where $t \geq 1$, of the affine space $AG_m(\mathbb{F}_2)$, which we will denote by $AG_{m,t}(\mathbb{F}_2)$, and in particular with the case of $t = m - 1$ (points and hyperplanes or $(m - 1)$-flats) and $t = m - 2$ (points and $(m - 2)$-flats).

For $F = \mathbb{F}_p$, where $p$ is a prime, the code $C_F = C_p(D)$ of the design $D$ over the finite field $F$ is the space spanned by the incidence vectors of the blocks over $F$. We take $F$ to be a prime field $\mathbb{F}_p$ where $p$ must divide the order of the design. If the incidence vector of a subset $Q$ of points is denoted by $v^Q$, then $C_F = \langle v^B \mid B \in B \rangle$, and is a subspace of $F^P$, the full vector space of functions from $P$ to $F$.

A linear code over $\mathbb{F}_q$ of length $n$, dimension $k$, and minimum weight $d$, is denoted by $[n, k, d]_q$. If $c$ is a codeword then the support of $c$, $\text{Supp}(c)$, is the set of non-zero coordinate positions of $c$, and the weight (or Hamming weight) of $c$, $\text{wt}(c)$, is the size of its support. A constant word in the code is a codeword all of whose non-zero coordinate entries are equal. The all-one vector $\mathbf{1}$ is the constant vector with all entries equal to 1. The value of $c$ at the coordinate position $P$ will be denoted by $c(P)$. An automorphism of a code $C$ is an isomorphism from $C$ to $C$.

Permutation decoding was introduced by MacWilliams [10] and Prange [12] and involves finding a set of automorphisms of a code called a PD-set. The method is described fully in MacWilliams and Sloane [11, Chapter 15] and Huffman [4, Section 8]. The concept of PD-sets was extended to $s$-PD-sets for $s$-error-correction in [6] and [8]:

**Definition 1** If $C$ is a $t$-error-correcting code with information set $I$ and check set $C$, then a PD-set for $C$ is a set $S$ of automorphisms of $C$ which is such that every $t$-set of coordinate positions is moved by at least one member of $S$ into the check positions $C$.

For $s \leq t$ an $s$-PD-set is a set $S$ of automorphisms of $C$ which is such that every $s$-set of coordinate positions is moved by at least one member of $S$ into $C$.

The efficiency of the algorithm for permutation decoding (see [4, Section 8], or [7, Section 2]) requires that the set $S$ is small; there is a combinatorial lower bound on its size due to Gordon [3] and Schönheim [13] (see [4] or [7]). A partial survey of known results concerning $s$-PD-sets for codes from designs and geometries can be found in [5] or at the website:

http://www.ces.clemson.edu/~keyj/ and, in particular,
3 Reed-Muller codes

We use the notation of [1, Chapter 5] or [2] for generalized Reed-Muller codes. Let $q = p^f$, where $p$ is a prime, and let $V$ be the vector space $\mathbb{F}_q^m$ of $m$-tuples, with standard basis. The codes will be $q$-ary codes with ambient space the function space $\mathbb{F}_q^V$, with the usual basis of characteristic functions of the vectors of $V$. We can denote the elements $f$ of $\mathbb{F}_q^V$ by functions of the $m$-variables denoting the coordinates of a variable vector in $V$; i.e. if $x = (x_1, x_2, \ldots, x_m) \in V$, then $f \in \mathbb{F}_q^V$ is given by $f = f(x_1, x_2, \ldots, x_m)$ and the $x_i$ take values in $\mathbb{F}_q$. Since $a^q = a$ for $a \in \mathbb{F}_q$, the polynomial functions can be reduced modulo $x_i^q - x_i$. Furthermore, every polynomial can be written uniquely as a linear combination of the $q^m$ monomial functions

$$M = \{x_1^{i_1}x_2^{i_2} \cdots x_m^{i_m} \mid 0 \leq i_k \leq q-1, \text{ for } 1 \leq k \leq m\}.$$

For any such monomial the degree $\rho$ is the total degree, i.e. $\rho = \sum_{k=1}^{m} i_k$ and clearly $0 \leq \rho \leq m(q-1)$.

The generalized Reed-Muller codes are defined as follows (see [1, Definition 5.4.1]):

**Definition 2** Let $V = \mathbb{F}_q^m$ be the vector space of $m$-tuples, for $m \geq 1$, over $\mathbb{F}_q$, where $q = p^f$ and $p$ is a prime. For any $\rho$ such that $0 \leq \rho \leq m(q-1)$, the $\rho^{th}$-order generalized Reed-Muller code $\mathcal{R}_{F_q}(\rho,m)$ is the subspace of $\mathbb{F}_q^V$ (with basis the characteristic functions of vectors in $V$) of all $m$-variable polynomial functions (reduced modulo $x_i^q - x_i$) of degree at most $\rho$. Thus

$$\mathcal{R}_{F_q}(\rho,m) = \langle x_1^{i_1}x_2^{i_2} \cdots x_m^{i_m} \mid 0 \leq i_k \leq q-1, \text{ for } 1 \leq k \leq m, \sum_{k=1}^{m} i_k \leq \rho \rangle.$$

These codes are thus codes of length $q^m$ and the codewords are obtained by evaluating the $m$-variable polynomials in the subspace at all the points of the vector space $V = \mathbb{F}_q^m$.

The code $\mathcal{R}_{F_p}((m-r)(p-1),m)$ is the $p$-ary code of the affine geometry design $AG_{m,r}(\mathbb{F}_p)$; see [1, Theorem 5.7.9].

The Reed-Muller codes are the codes $\mathcal{R}_{F_2}(r,m)$ and are usually written simply as $\mathcal{R}(r,m)$, where $0 \leq r \leq m$. The standard well-known facts concerning $\mathcal{R}(r,m)$ (see, for example, [1, Theorem 5.3.3]), can be summarized as:

**Result 1** For $0 \leq r \leq m$, $\mathcal{R}(r,m)$ is a $2^m, \binom{m}{r}, \binom{m}{r+1}, \cdots, \binom{m}{m}$ binary code. Furthermore, $\mathcal{R}(r,m) = C_{2^m}(AG_{m,m-r}(\mathbb{F}_2))$ and the minimum-weight vectors are the incidence vectors of the $(m-r)$-flats. The automorphism group of $\mathcal{R}(r,m)$ is the affine group $AGL_{m}(\mathbb{F}_2)$ for $0 < r < m-1$.

For permutation decoding, the following is Proposition 1 of [7] stated for generalized Reed-Muller codes:

**Result 2** Let $f_{n,m,q}$ denote the dimension and $d_{n,m,q}$ the minimum weight of $\mathcal{R}_{F_q}(n,m)$. If $s = \min\left(\lfloor (q^m-1)/f_{n,m,q}\rfloor, \lfloor (d_{n,m,q}-1)/2\rfloor\right)$, then the translation group $T_{m}(\mathbb{F}_q)$ is an $s$-PD-set for $\mathcal{R}_{F_q}(n,m)$. 

For the Reed-Muller codes this becomes:

**Result 3** For $0 \leq r \leq m$, the translation group $T_m (F_2)$ is an s-PD-set for $R(r, m)$, for

$$s = \min \left( \frac{(2^m - 1)}{\rho_r, m}, 2^{m-r} - 1 \right),$$

where $\rho_r, m = \binom{m}{r} + \binom{m}{r+1} + \cdots + \binom{m}{m}$.

These results hold for any information set for the code. As an illustration of Result 3, Table 1 shows the value of $s$ for which the translation group is an s-PD-set (of size $2^m$) for $R(1, m)$ or $R(2, m)$, and $4 \leq m \leq 16$, using any information set.

<table>
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<th>$m$</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
</tr>
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<tbody>
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<td>5</td>
<td>9</td>
<td>15</td>
<td>28</td>
<td>51</td>
<td>93</td>
<td>170</td>
<td>315</td>
<td>585</td>
<td>1092</td>
<td>2047</td>
<td>3855</td>
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<tr>
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<td>1</td>
<td>2</td>
<td>4</td>
<td>6</td>
<td>11</td>
<td>18</td>
<td>30</td>
<td>51</td>
<td>89</td>
<td>154</td>
<td>270</td>
<td>478</td>
</tr>
</tbody>
</table>

Table 1: Translation group as s-PD-set

We will use coding-theoretic terminology and notation for vectors in $V = F_2^m$; we do not expect that any confusion should arise with the vectors in the code $R(r, m)$ since we will not need to deal with the latter vectors in our search for PD-sets. Thus, using the standard basis $\{e_1, \ldots, e_m\}$ for $V = F_2^m$, and writing $e_0$ for $0 \in V$, for each $v = \sum_{i=1}^{m} \lambda_i e_i \in V$, let $wt(v)$ be the weight of $v$, i.e. the number of non-zero $\lambda_i$. The support of $v = \sum_{i=1}^{m} e_i \in V$ will be denoted by $\text{Supp}(v) = \{i_1, \ldots, i_r\}$. If $X \subseteq \{1, \ldots, m\}$, then $v(X)$ will denote the vector with support $X$, and if $X = \{i_1, \ldots, i_r\}$ we will write simply $v(i_1, \ldots, i_r)$, for convenience. (This contrasts with notation $v^X$ for codewords described in Section 2 above.)

Following the notation in [14], for $0 \leq i \leq m$, let

$$C_i = \{v \mid v \in V, \text{wt}(v) = i\}. \quad (1)$$

Let $f = x_{i_1} \cdots x_{i_r}$ be a monomial function of degree $r$. If $i < r$ and $v \in C_i$ then $f(v) = 0$. Also, if $i = r$, $v \in C_i$ and $v \neq e_{i_1} + \cdots + e_{i_r}$ then $f(v) = 0$. So, it is easily seen that

$$I_r = C_0 \cup C_1 \cup \cdots \cup C_r \quad (2)$$

is an information set for $R(r, m)$. (Alternatively, see [7, Corollary 2].)

The translation group $T_m (F_2)$ acts on $R(r, m)$ in the following way: for each $u \in V$, denote by $T_u$ the translation of $V$ given by $T_u : v \mapsto v + u$. This mapping acts on $R(r, m)$ by $f \mapsto f_u = f_{0}T_u$, i.e. $f_u(v) = f(u + v)$ for all $v \in V$.

## 4 s-PD-sets for $R(1, m)$

We now look for subsets of the translation group that will be s-PD-sets for $R(1, m)$ for some $s$. Using the notation from the previous section, let

$$A_m = \{T_u \mid u \in C_0 \cup C_1 \cup C_2 \cup C_m\}. \quad (3)$$

Then $|A_m| = \binom{m+1}{2} + 2 = \frac{1}{2} (m^2 + m + 4)$. We will use the information set $I = I_1$ with check set $C = V \setminus I$. We write $e = e_1 + e_2 + \cdots + e_m$ for the all-one vector of $V = F_2^m$.

In [14], the following result is proved.
Result 4 For $m \geq 5$, $A_m$ is a 4-PD-set of size $\binom{m+1}{2} + 2$ for $R(1, m)$ with respect to the information set $T$.

It is also conjectured in [14] that $A_{m}$ is a 5-PD-set for $R(1, m)$. This is true for $m \geq 6$ but not for $m = 5$. There are further results for PD-sets for punctured first-order Reed-Muller codes for small $m$ in [9].

We prove the following:

Proposition 1 If $m \geq 5$ then $A_{m}$ is a $(m-1)$-PD-set, but not an $m$-PD-set, for the $[2^{m}, m+1, 2^{m-1}]_{2}$ code $R(1, m)$ with respect to the information set $T$.

Proof: Let $S = \{0, e_{1}, \ldots, e_{m-3}, e_{m-2} + e_{m-1} + e_{m}, e_{m}\}$. It is immediate that $ST_{u} = \{u, e_{1} + u, \ldots, e_{m-3} + u, e_{m-2} + e_{m-1} + e_{m} + u, e + u\}$ has an element of weight at most 1 if $u \in C_{0} \cup C_{1} \cup C_{2} \cup C_{m}$. Hence, $ST_{\emptyset} \not\subseteq C$ for all $\emptyset \in A_{m}$, and so $A_{m}$ is not an $m$-PD-set for any $m$.

Now suppose that $S$ is an $(m-1)$-subset of $V$. We write $S_{i} = \{v \mid v \in S, wt(v) = i\} = S \cap C_{i}$ and $l_{j} = |S_{j}|$, $0 \leq i \leq m$. Then $m - 1 = \sum_{i=0}^{m} l_{i}$. For every choice of $S$ we need to find a translation $T_{u} \in A_{m}$ such that $ST_{u} \subseteq C$.

If $l_{m-1} + l_{m} = 0$, then $ST_{e} \subseteq C$. This includes the case $l_{1} = m - 1$. If $l_{m-1} + l_{m} = 1$ and $l_{1} = m - 2$, then $0 \notin S$ and $C_{1} \backslash S_{1} = \{e_{i}, e_{j}\}$ for some $i$ and $j$. In this case, $ST_{e} \subseteq C$.

Now, suppose that $l_{m-1} + l_{m} \geq 1$ and $l_{1} \leq m - 3$. Let $n = m - l_{1}$. Thus, $n \geq 3$. By relabelling the elements of the basis of $V$, we may suppose that $C_{1} \backslash S_{1} = \{e_{1}, \ldots, e_{n}\}$.

Since $m \geq 5$, $m - 1 \geq n_{0} + l_{1} + l_{2} + l_{3} + l_{m-1} + l_{m} \geq m - n + 1 + l_{0} + l_{2} + l_{3}$. Hence, $l_{2} + l_{3} \leq n - 2 - l_{0}$. Note that $l_{0} = 0$ or 1.

Let $[1, n] = \{1, \ldots, n\}$. For $i, j \in [1, n]$ with $i < j$, we define: (i) $a_{i,j}$ to be 1 if $e_{i} + e_{j} \in S$ and 0 otherwise; (ii) $b_{i,j}$ to be the number of $k$ with $1 \leq k \leq n$ and $\{i, j, k\}$ a 3-set for which $e_{i} + e_{j} + e_{k} \in S$; (iii) $c_{i,j}$ to be the number of $k$ with $k > n$ and $\{i, j, k\}$ a 3-set for which $e_{i} + e_{j} + e_{k} \in S$. The sum $l_{2}^{*} = \sum_{1 \leq i < j \leq n} a_{i,j}$ counts the number of elements in $S_{2}$ of the form $e_{i} + e_{j}$ with $1 \leq i, j \leq n$. The sum $l_{3}^{*} = \frac{1}{3} \sum_{1 \leq i < j \leq n} b_{i,j}$ counts the number of elements in $S_{3}$ of the form $e_{i} + e_{j} + e_{k}$ with $1 \leq i, j, k \leq n$. The sum $\sum_{1 \leq i < j \leq n} c_{i,j}$ counts the number of elements in $S_{3}$ of the form $e_{i} + e_{j} + e_{k}$ with exactly two of the indices $i, j, k$ in $[1, n]$. Hence, $l_{3}^{*} = \sum_{1 \leq i < j \leq n} \left(\frac{1}{3} b_{i,j} + c_{i,j}\right)$ counts the number of elements in $S_{3}$ of the form $e_{i} + e_{j} + e_{k}$ with $\{i, j, k\} \cap [1, n] \geq 2$.

Hence, writing $f_{i,j} = a_{i,j} + \frac{1}{3} b_{i,j} + c_{i,j}$, we get

$$\sum_{1 \leq i < j \leq n} f_{i,j} = l_{2}^{*} + l_{3}^{*} \leq l_{2} + l_{3} \leq n - 2 - l_{0}$$ \hspace{1cm} (4)

Suppose that $f_{i,j} > 0$ for all 2-sets $i, j$ with $1 \leq i < j \leq n$. Then, the left hand side of equation (4) is at least $\frac{1}{3} \binom{n}{2}$. Moreover, $\frac{1}{3} \binom{n}{2} - (n - 2) = \frac{1}{3} \left(n - \frac{3}{2}\right) \geq 0$ for $n \geq 3$. Hence, the inequalities in (4) are equalities, $l_{0} = 0$ and $n = 3$ or 4. Also, for every pair $\{i, j\}$ in $[1, n]$, either $e_{i} + e_{j} \in S_{2}$ or $e_{i} + e_{j} + e_{k} \in S_{3}$ for some $k$ different from $i$ and $j$.

If $n = 3$, then $l_{2}^{*} = l_{3}^{*} = 2 + l_{3} = 1$. Hence, $l_{2}^{*} = 0$, $l_{3}^{*} = 1$, $S_{2} = \emptyset$ and $S_{3} = \{e_{1} + e_{2} + e_{3}\}$. But then $ST_{e_{3}} \subseteq C$. 

If $n = 4$, then $l_2 + l_3 = 2$. By the condition $f_{i,j} > 0$ for every pair $i,j$ in $[1,4]$, all six pairs must occur in the support of vectors of weight 2 or 3. However, since at most five pairs can occur in two vectors of weight 2 or 3, this case cannot occur.

Otherwise, we can find $i, j \in [1, n]$ with $i < j$ and $f_{i,j} = 0$. Hence, $a_{i,j} = b_{i,j} = c_{i,j} = 0$. Let $u = e_i + e_j$. Then neither $u$ nor any $u + e_i$, with $1 \leq l \leq m$, $l \neq i, j$, is in $S$.

So, if $v \in S_2 \cup S_3$ we get $w_t(v + u) \geq 2$ by the choice of $u$. If $v \in S_i$ with $i \geq 4$, $w_t(v + u) \geq i - 2 \geq 2$. If $v \in S_1$, we have $v = e_k$ with $k > n$ and consequently $w_t(v + u) = 3$. Finally, $w_t(0 + u) = 2$. Hence, $ST_u \subseteq C$. ■

We now improve on this, but we need to increase the set of translations. Thus let

$$B_m = \{T_u \mid u \in C_0 \cup C_1 \cup C_2 \cup C_3 \cup C_m\}. \quad (5)$$

Then $|B_m| = 2 + m + \binom{m}{2} + \binom{m}{3} = \frac{1}{6}(m^3 + 5m + 12)$.

**Proposition 2** If $m \geq 6$ then $B_m$ is an $(m + 1)$-PD-set for the $[2^m, m + 1, 2^{m - 1}]_2$ code $R(1, m)$ with respect to the information set $I$.

**Proof:** Use the notation of Proposition 1. The check set $C$ corresponding to $I$ consists of all vectors of weight at least 2. Here $S$ is an $(m + 1)$-subset of $V$, and $m + 1 = \sum_{i=0}^{m} l_i$.

We need to show that for every choice of $S$, there is a translation $T_u \in B_m$ such that $ST_u \subseteq C$. As before, $S_i = S \cap C_i$ for $0 \leq i \leq m$.

If $l_{m-1} + l_m = 0$, then $ST_e \subseteq C$. So suppose $l_{m-1} + l_m \geq 1$. If $l_1 = m$ and $S \setminus C_1 = \{u\}$ where $w_t(u) \geq m - 1$, then $T_{e_1 + e_2 + e_3}$ will work, since $m \geq 6$. If $l_1 = m - 1$ and $C_1 \setminus S_1 = \{e_i\}$ then $T_{e_i}$ will work unless the remaining element in $S$ is $0$ or $e_i + e_j$, for some $j \neq i$. In either case $T_{e_i + e_k + e_l}$, where $k, l \neq j$, will do.

Thus we take $l_1 \leq m - 2$. As in the proof of Proposition 1, let $n = m - l_1$ where $n \geq 2$.

Since $m \geq 6$, $m + 1 \geq l_0 + l_1 + l_2 + l_3 + l_{m-1} + l_m \geq m - n + 1 + l_0 + l_2 + l_3$. Hence, $l_2 + l_3 \leq n - l_0$ where $l_0 = 0$ or 1.

By relabelling the elements of the basis for $V$, we may suppose that $C_1 \setminus S_1 = \{e_1, \ldots, e_n\}$. We continue with the notation introduced in the proof of Proposition 1. We can write $S = \{0, e_{n+1}, \ldots, e_m, u_1, \ldots, u_n\}$ or $S = \{e_{n+1}, \ldots, e_m, u_1, \ldots, u_m, u_{n+1}\}$, according as $l_0 = 1$ or 0, where the first $l_2' + l_3'$ of the $u_i$’s are the elements of $S_2 \cup S_3$ meeting $[1, n]$ in at least two points, the next $l_2 + l_3 - l_2' - l_3'$ of the $u_i$’s are the remaining elements of $S_2 \cup S_3$, and the remaining $u_i$’s, of which there is at least one, have weight at least 4.

Also, $w_t(u_n) \geq m - 1 \geq 5$ or $w_t(u_{n+1}) \geq m - 1 \geq 5$ according as $l_0 = 1$ or 0.

Arguing as in the proof of Proposition 1, if $f_{i,j} > 0$ for all pairs $i, j \in [1, n]$, then $\frac{1}{3} \binom{n}{2} \leq n - 1$ if $l_0 = 1$, and $\frac{1}{3} \binom{n}{2} \leq n$ if $l_0 = 0$. We deal with these two cases separately. Note that $f_{i,j} > 0$ implies that $l_2 + l_3 \geq l_2' + l_3' > 0$.

1. $l_0 = 1$. Then $\frac{1}{3} \binom{n}{2} \leq l_2' + l_3' \leq l_2 + l_3 \leq n - 1$. In particular, $2 \leq n \leq 6$.

The number $l_2' + l_3'$ of 2-sets and 3-sets in $[1, n]$ needed to contain all 2-sets is at least $n - 1$ for $n = 2$ or $4 \leq n \leq 5$, at least 1 if $n = 3$ and at least 6 for $n = 6$.

Hence, the case $n = 6$ cannot occur and, for $n = 2$, 4 and 5, $l_2' = l_2$, $l_3' = l_3$. Moreover, for $n = 4$ and 5, $l_3' \geq n - 2$. 

For \( n = 2, 4 \) and \( 5 \), at most one of the elements \( u_1, \ldots, u_{n-1} \) has a support meeting \([1, n]\) in 2 points. Hence, \( T_{v(1, n+1, m)} \) will map \( S \) into \( C \) unless \( n = 5 \) and \( m = 6 \). In this case, each of \( u_1, u_2, u_3 \) and \( u_4 \) must have weight 3 and their supports must lie in \([1, 5]\). Hence, \( T_{v(1, 2, 6)} \) will map \( S \) into \( C \).

If \( n = 3 \), then we have \( u_1 = e_1 + e_2 + e_3 \). If possible, choose \( i \in [1, 3] \setminus Supp(u_2) \) and let \( v = v(i, 4, 5) \). This will certainly be the case if \( wt(u_2) \leq 3 \). If \( wt(u_2) = 4 \) and \([1, 3] \subseteq Supp(u_2) \), let \( v = v(i, j, k) \) with \( j, k \in [4, m] \setminus Supp(u_2) \) and \( j \neq k \). If \( wt(u_2) \geq 5 \), let \( v = v(1, 4, 5) \). In all cases, \( T_v \) will map \( S \) into \( C \).

2. \( l_0 = 0 \). Then \( \frac{1}{2}\binom{n}{2} \leq l_2^* + l_3^* \leq l_2 + l_3 \leq n \). In particular, \( 2 \leq n \leq 7 \). Also, if there is an \( i \in [1, n] \) which is not in the support of any \( u_j \) of weight 2 then \( T_{v_i} \) will map \( S \) into \( C \). So, we may suppose that every \( i \in [1, n] \) is in the support of some \( u_j \) of weight 2. We will refer to this as assumption \((*)\).

As for the case \( l_0 = 1 \), we see that \( l_2^* + l_3^* \geq 1, 1, 3, 4 \) or 6 according as \( n = 2, 3, 4, 5 \) or 6. Moreover, \( l_3^* \geq 2, 3 \) or 6 according as \( n = 4, 5 \) or 6. Additionally, when \( n = 7 \) we see that \( l_2^* + l_3^* \geq 7 \) and \( l_3^* \geq 7 \).

(i) \( n = 2 \). If \( |Supp(u_1) \cup Supp(u_2)| \leq m - 2 \), let \( v = v(i, j, k) \) where \( j, k \notin Supp(u_1) \cup Supp(u_2) \) and \( i \neq j \) and \( k \). Otherwise, \( |Supp(u_1) \cup Supp(u_2)| \geq m - 1 \geq 5 \). Hence, \( wt(u_1) \) and \( wt(u_2) \) are not both 2. By assumption \((*)\), \( u_1 = e_1 + e_2 \).

If possible, choose \( i \in Supp(u_1) \setminus Supp(u_2) \) and \( j, k \in Supp(u_2) \setminus Supp(u_1) \) with \( j \neq k \), and let \( v = v(i, j, k) \). If this is not possible, then \( Supp(u_1) \subseteq Supp(u_2) \) and \( |Supp(u_2)| \geq m - 1 \geq 5 \). We can then choose distinct \( i, j, k \in Supp(u_2) \setminus Supp(u_1) \) and let \( v = v(i, j, k) \).

In all cases, \( T_v \) maps \( S \) into \( C \).

(ii) \( n = 3 \). Suppose first that \( e_1 + e_2 + e_3 = u_1 \in S \). By assumption \((*)\), \( wt(u_i) = 2 \) and \( Supp(u_i) \subseteq [1, 3] \cup \{j\} \) for some \( j \in [4, m] \) and for \( i = 2 \) and \( 3 \). Let \( v = v(1, k, l) \), where \( k, l \in [4, m] \setminus \{j\} \) and \( k \neq l \).

If \( e_1 + e_2 + e_3 \notin S \) then \( l_2^* + l_3^* = 3 \) and \( u_1 = e_1 + e_2 + \delta_1 e_j, u_2 = e_2 + e_3 + \delta_2 e_k \) and \( u_3 = e_1 + e_3 + \delta_3 e_l \), where \( \delta_i \in \{0, 1\} \) for \( i \in [1, 3] \) and \( j, k, l \in [4, m] \) but not necessarily distinct. Let \( v = v(4, 5, 6) \).

In all cases, \( T_v \) maps \( S \) into \( C \).

(iii) \( n = 4 \) or 5. We may now assume that \( wt(u_1) = 2 \).

At most two of the \( u_i, 1 \leq i \leq n \), have supports meeting \([1, n]\) in sets of size at most 2. By \((*)\), we must have \( n = 4, wt(u_2) = 2 \) and \( Supp(u_1) \cup Supp(u_2) = [1, 4] \). Then \( l_2^* + l_3^* = l_2 + l_3 = 4 \) and \( wt(u_3) = 3 \) and \( Supp(u_i) \subseteq [1, 4] \) for \( i = 3 \) and 4. Let \( v = v(1, 4, 5) \). Then \( T_v \) maps \( S \) into \( C \).

(iv) \( n = 6 \) or 7. Here \( l_2^* + l_3^* = n \) and all \( n \) elements of \( S_2 \cup S_3 \) have weight 3. This is excluded by assumption \((*)\).

Thus there is a pair \( i, j \) for which \( f_{i,j} = 0 \) and we can complete the proof as in Proposition 1. ■
5 s-PD-sets for \( \mathcal{R}(2, m) \)

We now adapt the method of proof of Propositions 1 and 2 to establish the following proposition for \( \mathcal{R}(2, m) \). Here the information set is \( \mathcal{I} = \mathcal{I}_2 \) and the check set is \( \mathcal{C} = V \setminus \mathcal{I} \); the latter consists of all vectors of weight at least 3. Other notation is as in Section 4.

**Proposition 3** If \( m \geq 8 \) then \( B_m \) is an \((m-3)\)-PD-set for the \([2^m, 1 + m + \binom{m}{2}, 2^{m-2}]_2\) code \( \mathcal{R}(2, m) \) with respect to the information set \( \mathcal{I}_2 \).

**Proof:** We first observe that \( B_m \) is not an \((m-2)\)-PD-set, since the \((m-2)\)-set \( S = \{e_1 + e_2 + e_3 + e_4, e_5, \ldots, e_m, e\} \) is not mapped into \( \mathcal{C} \) by translation with any element of \( B_m \).

Now let \( S \) be a set of size \((m-3)\) in \( V \). As before, \( S_1 = S \cap C_4 \) and \( l_1 = |S_1| \), for \( 0 \leq i \leq m \). Thus \( m - 3 = \sum_{i=0}^m l_i \). We let \( n = m - l_1 \) and arrange the notation so that \( C_1 \setminus S_1 = \{e_1, \ldots, e_n\} \). We have to show that there is an element \( v \in B_m \) so that \( ST_v \subseteq \mathcal{C} \).

If \( l_{m-2} + l_{m-1} + l_m = 0 \), then we may take \( v = e \). For the rest of the proof, we assume that \( l_{m-2} + l_{m-1} + l_m \geq 1 \). If \( l_1 = m - 4 \), then \( l_0 = 0 \) and we may take \( v = e_1 + e_2 \). Thus, we may assume that \( l_1 \leq m - 5 \), that is, \( n \geq 5 \). Since \( m \geq 8 \), \( m - 3 \geq l_0 + l_1 + l_2 + l_3 + l_4 + l_5 + l_{m-2} + l_{m-1} + l_m \geq m - n + 1 + l_0 + l_2 + l_3 + l_4 + l_5 \).

Hence, \( l_2 + l_3 + l_4 + l_5 \leq n - 4 \).

Now we define a collection of functions defined on the triples \( \{i, j, k\} \) of \( [1, n] \). To simplify notation we will suppose that \( 1 \leq i < j < k \leq n \). Then (i) \( a_{i,j,k} \) is the number of triples \( \{i', j', k'\} \subseteq [1, m] \) with \( v(i', j', k') \in S_2 \) and \( i', j' \in \{i, j, k\} \); (ii) \( b_{i,j,k}^{(p)} \) is the number of triples \( \{i', j', k'\} \subseteq [1, m] \) with \( v(i', j', k') \in S_3 \) and \( \{i', j', k'\} \nsubseteq \{i, j, k\} \) = \( p = \|\{i', j', k'\} \cap [1, n]\| \), for \( p = 2 \) and 3; (iii) \( c_{i,j,k}^{(p)} \) is the number of quadruples \( \{i', j', k', l'\} \subseteq [1, m] \) with \( v(i', j', k', l') \in S_4 \), \( \{i', j', k', l'\} \subseteq \{i, j, k, n\} \) = \( p = 3 \) or 4; (iv) \( d_{i,j,k}^{(p)} \) is the number of quintuples \( \{i', j', k', l', m'\} \subseteq [1, m] \) with \( v(i', j', k', l', m') \in S_5 \), \( \{i', j', k', l', m'\} \subseteq \{i, j, k, n\} \) = \( p = 3 \) or 4.

If \( l_2' \) is the number of pairs \( \{i', j'\} \subseteq [1, n] \) with \( v(i', j') \in S_2 \), then \( \sum_{1 \leq i < j < n} a_{i,j,k} = l_2'(n - 2) \).

Clearly, \( \sum_{1 \leq i < j < k \leq n} b_{i,j,k}^{(3)} \) is the number \( l_2' \) of elements of \( S_3 \) with support in \([1, n]\). If \( l_3'' \) denote the number of elements of \( S_3 \) whose support meets \([1, n]\) in a set of size 2, then \( \sum_{1 \leq i < j < k \leq n} b_{i,j,k}^{(2)} = l_3''(n - 2) \).

If \( l_4' \) and \( l_4'' \) denote the numbers of elements of \( S_4 \) whose support meets \([1, n]\) in sets of size 3 and 4, respectively, then \( \sum_{1 \leq i < j < k \leq n} c_{i,j,k}^{(3)} = l_4' \) and \( \sum_{1 \leq i < j < k \leq n} c_{i,j,k}^{(4)} = 4l_4'' \).

If \( l_5' \), \( l_5'' \) and \( l_5''' \) denote the numbers of elements of \( S_5 \) whose support meets \([1, n]\) in sets of size 3, 4 and 5, respectively, then \( \sum_{1 \leq i < j < k \leq n} d_{i,j,k}^{(3)} = l_5' \), \( \sum_{1 \leq i < j < k \leq n} d_{i,j,k}^{(4)} = 4l_5'' \), and \( \sum_{1 \leq i < j < k \leq n} d_{i,j,k}^{(5)} = 10l_5''' \).
For each triple $i, j, k$ with $1 \leq i < j < k \leq n$, define

$$f_{i,j,k} = \frac{1}{n-2} a_{i,j,k} + \frac{1}{n-2} b_{i,j,k}^{(2)} + \frac{1}{n-2} b_{i,j,k}^{(3)} + c_{i,j,k}^{(3)} + \frac{1}{4} f_{i,j,k}^{(4)} + \frac{1}{4} f_{i,j,k}^{(5)} + \frac{1}{10} d_{i,j,k}^{(5)}.$$ 

Since $l_2' \leq l_2, l_3' + l_3'' \leq l_3, l_4' + l_4'' \leq l_4$ and $l_5' + l_5'' + l_5''' \leq l_5$,

$$\sum_{1 \leq i < j < k \leq n} f_{i,j,k} \leq l_2 + l_3 + l_4 + l_5 \leq n - 4 - l_0 \quad (6)$$

We will show that there is a triple $i, j, k$ with $1 \leq i < j < k \leq n$ such that $f_{i,j,k} = 0$, or find an element $v \in B_m$ with $ST_v \in \mathcal{C}$. Suppose that $f_{i,j,k} > 0$ for all triples $i, j, k$ with $1 \leq i < j < k \leq n$. Then, the left hand side of equation (6) is at least $\frac{1}{10} \binom{n}{3}$ if $n < 12$ and at least $\frac{1}{10} \binom{n}{3}$ if $n \geq 12$. Since $\frac{1}{n-2} \binom{n}{3} = \frac{n(n-1)}{6} > n - 4$ if $n \geq 12$, we must have $n < 12$. Also, $\frac{1}{10} \binom{n}{3} > n - 4$ if $7 \leq n \leq 11$. So we must have $n = 5$ or 6.

If $n = 5$ or 6, $\frac{1}{10} \binom{n}{3} = n - 4$ so that $l_0 = 0$ and all terms in the definition of $f_{i,j,k}$ are 0 with the exception of $\frac{1}{10} d_{i,j,k}^{(5)}$ which is $\frac{1}{10}$. Then $d_{i,j,k}^{(5)} = 1$ for every triple $i, j, k$ in $[1, n]$ implies that $l_5 \geq 1$ if $n = 5$ and $l_5 \geq 4$ if $n = 6$, since every triple in $[1, n]$ is in the support of an element of $S_5$. The latter is impossible since $l_5 \leq n - 4 - l_0 = 2$. When $n = 5$, for each triple $i, j, k$ with $1 \leq i < j < k \leq n$, $a_{i,j,k} = 0$, $b_{i,j,k}^{(p)} = 0$ if $p = 2$ and 3, and $c_{i,j,k}^{(p)} = 0$ if $p = 3$ and 4. Thus no element of $S_2$ has support in $[1, n]$, no element of $S_3$ has a support meeting $[1, n]$ in more than one point and no element of $S_4$ has a support meeting $[1, n]$ in more than two points. Since $l_4 \leq n - 4 - l_5$ we have $l_4 = 0$. We may choose $v = v(1, 2)$ and then $ST_v \in \mathcal{C}$.

It remains to deal with those cases in which there is a triple $i, j, k$ with $1 \leq i < j < k \leq n$ such that $f_{i,j,k} = 0$. For such a triple, (i) it contains the support of no element of $S_2$, (ii) it does not meet the support of any element of $S_3$ in more than one point, and (iii) it does not meet the support of any element of $S_4 \cup S_5$ in more than two points. Hence, if we set $v = v(i, j, k)$ then $ST_v \in \mathcal{C}$. This completes the proof. \[\square\]

Note: We cannot take $m = 7$ in Proposition 3 since the set $S = \{0, e_1, e_2, e_3 + e_4 + e_5 + e_6 + e_7\}$ cannot be moved into $\mathcal{C}$ by $B_7$.

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References


