Calculations of the minimal perimeter for N deformable bubbles of equal area confined in an equilateral triangle

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Abstract

Candidates to the least perimeter partition of an equilateral triangle into N planar connected regions are calculated for $N \leq 45$. A Voronoi construction is used to randomly create the candidates and then the perimeter of each is found with the Surface Evolver. The optimal configuration for each N has no more than one defect pair, and its location is determined by the proximity of N to a triangular number, allowing a prediction to be made for the optimal structure of partitions for larger N.

1 Introduction

The surface energy of a two-dimensional foam is simply its perimeter multiplied by surface tension (Weaire and Hutzler, 1999). A foam attains a local minimum of this perimeter, subject to the constraint of fixed bubble volumes. Here, we seek the arrangement of bubbles that gives the global minimum.

The local structure of perimeter-minimizing bubble clusters is well defined: perimeter minimization implies Plateau's rules (Plateau, 1873; Taylor, 1976): three and only three edges meet at a point at 120°. The Laplace Law relating pressure difference and curvatures gives the further condition that each edge is a circular arc. These conditions are augmented in the present case by the rule that edges meet the bounding walls at 90°.

For bubbles filling the plane, the hexagonal honeycomb is optimal in this sense (Hales, 2001). For finite clusters, the problem is related in that bubbles far from the periphery of the cluster are likely to be hexagonal. For small numbers of bubbles, $N \leq 3$, there are various proofs of optimality for free clusters (in R^n), and clusters confined in disks; see e.g. Morgan (2000); Wichiramala (2004); Cañete and Ritore (2004). For larger N, and different confining geometries, there are only conjectures; see e.g. Bleicher (1987); Alfaro et al. (1990); Cox et al. (2003); Cox (2006); Baumann (2007). The latter are usually based upon the assumption that in the global minumun each bubble consists of a single component; we use this assumption of connectedness for the triangular confinement considered here.

In this paper we impose the periphery of the cluster to be an equilateral triangle, determine candidates to the least perimeter arrangement, and examine the influence of the periphery in creating deviations of the cluster from the regular hexagonal honeycomb. That is, we seek the least perimeter partition of an equilateral triangle into N cells of equal area, equivalent to the energetic groundstate for N monodisperse bubbles or the optimal packing of equal-area objects. We examine values of N up to 45 and record the least perimeter, the number of peripheral bubbles N_p and the configuration that realizes the least perimeter.

Triangular numbers of the form $N_T = \frac{1}{2}i(i+1)$, for positive integers i, are "magic" numbers. That is, for values of $N = N_T$ the cluster formed by cutting a triangular segment from a hexagonal honeycomb and allowing it to deform slightly to satisfy area constraints provides the global minimum. For other N there are defects in the optimal solution, by which we mean bubbles in the bulk that do not have six sides, bubbles that touch a wall and don't have five sides, and bubbles in the corners of the triangle that don't have four sides. This can be made general with the idea of topological charge (Graner et al., 2001): bubbles have a charge $q = 6 - n - b$, where n is the number of sides (including those that form the external boundary of the triangle) and b is the number of boundaries with which the bubble is in contact. Thus hexagons in the bulk have zero charge. Cox et al. (2003) found that in the free cluster case, positive and negative charges tended to be associated.

2 Method

We consider an equilateral triangle of side-length $L = 1$, containing $N \leq 45$ bubbles of area $A = \sqrt{3}L^2/(4N)$. We report the internal perimeter P, or cut-length (i.e. the total perimeter minus three).

For small $N \leq 6$, we enumerated all possible (connected) candidates to find the least perimeter one.

For larger N, we use a method based upon a numerical computation of the optimal arrangement of particles with a $1/r^2$ inter-particle potential and a harmonic confining potential (Press et al., 1993). The latter is deformed to be triangular in shape. For each N many different arrangements of the particles are computed, based upon random initial placements. The resulting pattern is used as the basis for a a Voronoi partition (Barber et al., 1996), and then the resulting cellular structure is imported into the Surface Evolver (Brakke, 1992), area constraints applied, and then converged to a minimum of perimeter. Any short edges are altered, through T1 neighbour switching processes (Weaire and Hutzler, 1999), to seek better nearby minima. This process becomes slower as N increases, and even though for $N > 20$, about 10,000 candidates are tried, and several thousand found, for each N, likely-looking minima are not always apparent. We therefore supplement this automated procedure for $N > 30$ with manipulations of the existing best candidate, based upon the intuition built up for smaller N.

3 Results

Our candidate configurations are shown in figure 1, and the perimeters plotted in figure 2 and tabulated in Table 1.

For small N, simple analytic calculations can be used to check the results. We find the same candidates for $N = 1,2,3$ and 6 as those found by Bleicher (1987) and give better ones for $N = 4$ and $N = 5$. To the best of our knowledge, no candidates have been given for higher N.

We make the following remarks about the candidate configurations:

1. It is never optimal to have an edge emanating from an apex of the triangle. We suspect that this could be proven rigorously, at least in the equilateral case.

2. For the triangular values, $N = 1,3,6,10,\ldots$, the optimal candidate consists of part of a hexagonal honeycomb, albeit a deformed one that accommodates the area constraints. Based upon

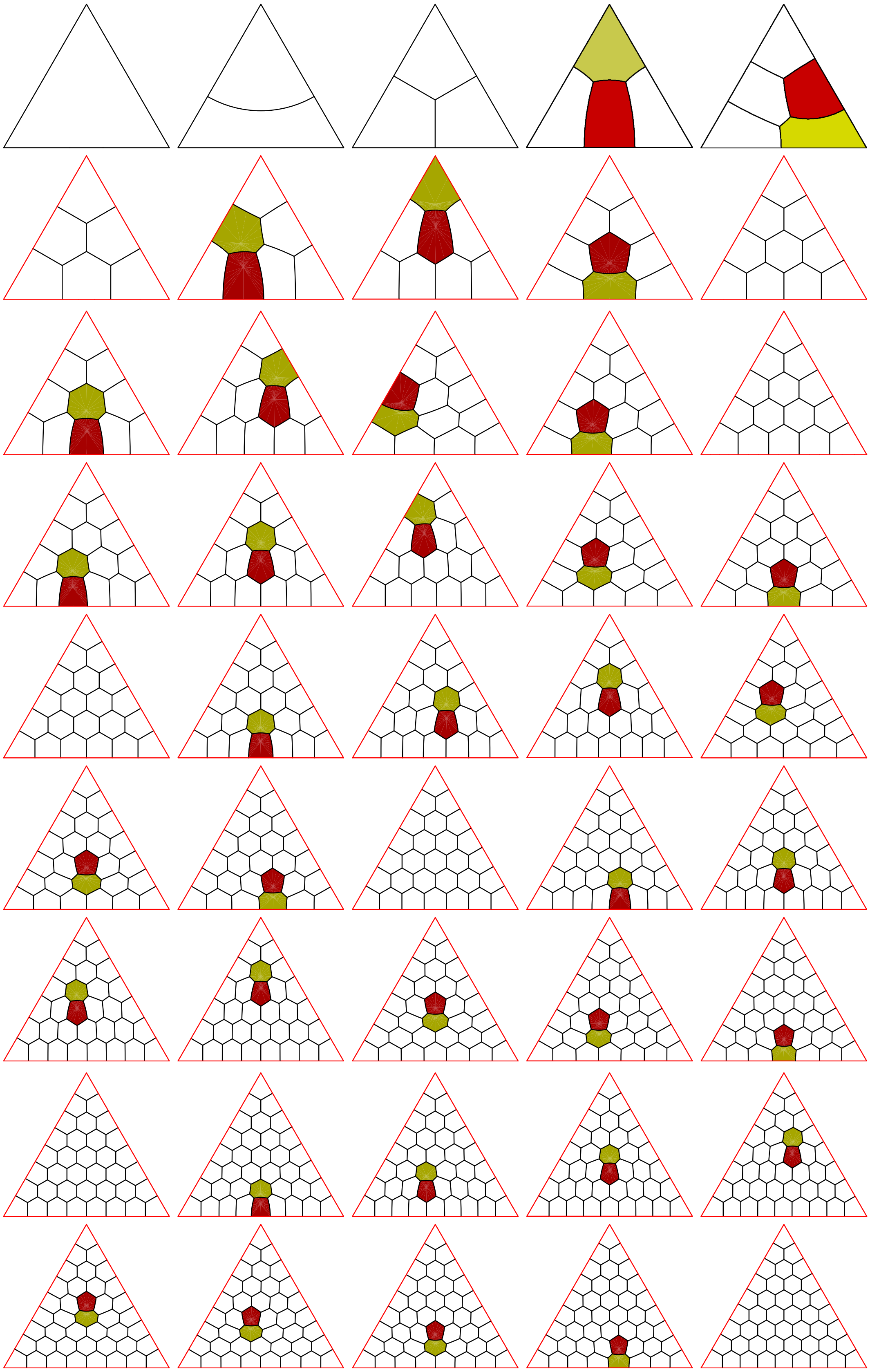

Figure 1: Candidate solutions for the least line length configuration of N bubbles within an equilateral triangle.

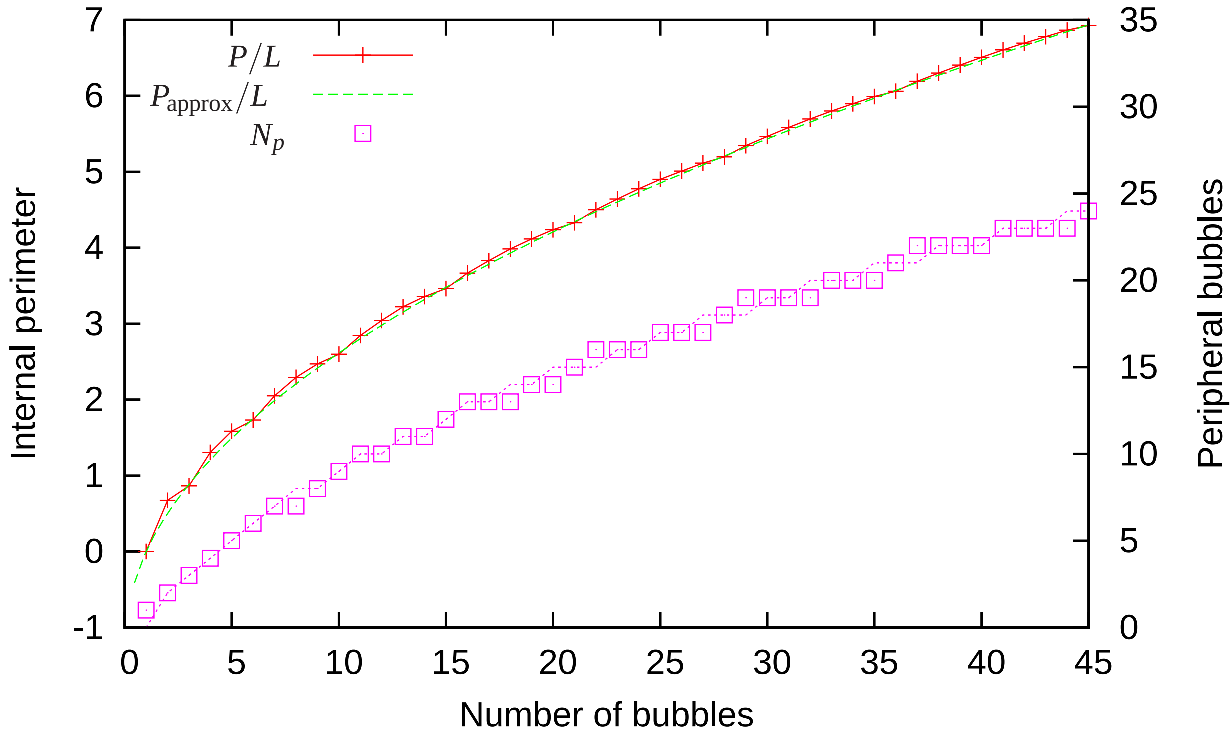

Figure 2: The perimeter P/L and the number of peripheral bubbles N_p of the conjectured least line length configuration. On close inspection it is clear that for N a triangular number the perimeter is particularly low. An approximate formula (2) based upon a perfect hexagonal honeycomb works well for all N. N_p increases monotonically, and is well fitted by the formula (1) for triangular configurations.

N	P/L	N_p	N	P/L	N_p	N	P/L	N_p
1	0.000000	1	16	3.666221	13	31	5.584035	19
2	0.673387	2	17	3.829858	13	32	5.697594	19
3	0.866025	3	18	3.986210	13	33	5.800204	20
4	1.305113	4	19	4.117581	14	34	5.897554	20
5	1.584560	5	20	4.238181	14	35	5.990828	20
6	1.732051	6	21	4.330127	15	36	6.062178	21
7	2.049893	7	22	4.501126	16	37	6.193138	22
8	2.294978	7	23	4.641201	16	38	6.300656	22
9	2.469856	8	24	4.775098	16	39	6.405179	22
10	2.598076	9	25	4.900145	17	40	6.507262	22
11	2.844104	10	26	5.010126	17	41	6.604324	23
12	3.043403	10	27	5.115975	17	42	6.693674	23
13	3.223179	11	28	5.196152	18	43	6.780117	23
14	3.357816	11	29	5.344561	19	44	6.864152	23
15	3.464102	12	30	5.465966	19	45	6.928203	24

Table 1: Perimeter P/L and the number N_p of peripheral bubbles of the minimal candidates found here.

the number of peripheral bubbles in this defect-free case, we propose the following formula for N_p:

$$N_p = \left[\frac{3}{2}\left(\sqrt{1+8N_T}-3\right)\right], \tag{1}$$

where [] denotes the nearest integer. This expression is shown on figure 2 and fits the data well, except (at higher N) on either side of the triangular values where N_p changes more steeply.

3. We also provide an approximate formula for the perimeter[1], based upon the idea that each bubble in the bulk attains its optimal hexagonal shape and that each wall bubble (except for those in the corners, which are not significant in this approximation) attains the optimal stretched-half hexagon shape that accommodates the area constraint. The resulting expression is

$$P_{\text{approx}} = (3N-2N_p-3)\sqrt{\frac{2A}{3\sqrt{3}}} + \frac{5N_p}{6}\sqrt{\frac{\sqrt{3}A}{2}}, \tag{2}$$

in which N_p is given by (1), shown on figure 2.

4. For all other N there is *exactly* one pair of defects, i.e. one bubble with positive topological charge adjacent to one bubble with negative topological charge. This provides a startling realization of the conjecture of Cox et al. (2003) that defects should associate.

5. The position of the defect pair depends upon the proximity of N to a triangular number. If N is of the form $N_T \pm 1$, then the defect pair will touch the boundary. Iif N is of the form $N_T + 1$ then the positive charge will be against the boundary while if N is of the form $N_T - 1$, the negative charge will touch the boundary. If N is two or more away from N_T, the defect pair will move into the bulk, keeping the same orientation of the positive-negative charge.

6. This suggests a recipe to generate the optimal candidate for each value of N, given the optimal defect-free candidate for the nearest N_T, as follows. Find the closest value of N_T to N, and if there are two, choose the lower. Consider each defect-free configuration for N_T as a stack of rows of bubbles. Then add (or subtract if $N < N_T$) bubbles to/from the middle of successive rows, starting from the row along the base of the triangle, until the number of bubbles reaches N. Then re-converge to equilibrium, allowing neighbour switching events on short edges where required.

4 Conclusion

We have found candidates to the minimal perimeter of partitions of an equilateral triangle into N regions of equal area. Equivalently, we have found the global energetic groundstate of a two-dimensional foam confined within a triangular boundary. We conjecture that optimal partitions for other N can be found from the nearest "magic" cluster, i.e. for N a triangular number, by adding/subtracting bubbles in successive layers from the wall.

[1]E. Flikkema pointed out to us that, based upon the perimeter for the $N = 3$ case, the expression $P_T/L = N_p/\sqrt{12}$ appears to give the perimeter of all triangular numbers, and thus provides a good lower bound for the perimeter for all N.

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