ph260 Theoretical Physics 2 — workshop 3 — solutions

1. Solving linear higher-order ODEs.
Solve the following second and third order ODEs.

a. $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x} - 2y = 0; \qquad y(0) = 12; \qquad y(12) = 0.$
Result: $y \approx 12\mathrm{e}^{-2x}$

b. $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 5\frac{\mathrm{d}y}{\mathrm{d}x} + 6y = 0; \qquad y(1) = y(-1); \qquad y(0) = 1.$
Result: $y \approx -0.57\mathrm{e}^{-3x} + 1.57\mathrm{e}^{-2x}$

c. $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 6\frac{\mathrm{d}y}{\mathrm{d}x} + 9y = 12\mathrm{e}^{-x}; \qquad y(-1) = 30; \qquad y(0) = 3.$
First solve the complementary homogeneous equation.
Both roots of its characteristic polynomial are identical - hence multiply one of the terms by x.
Complementary function: $y_c = c_1\mathrm{e}^{-3x} + c_2x\mathrm{e}^{-3x}$.
Particular solution is likely to be of form $y_p = a\mathrm{e}^{-x}$. *Substitute into DE to find a.*
Particular function: $y_p = 3\mathrm{e}^{-x}$.
Full solution, with BC applied: $y(x) = y_c(x) + y_p(x) \approx -1.09x\mathrm{e}^{-3x} + 3\mathrm{e}^{-x}$.

d. $\frac{\mathrm{d}^3y}{\mathrm{d}x^3} - 3\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 9\frac{\mathrm{d}y}{\mathrm{d}x} - 5y = 0.$
First root can be guessed as $u_1 = -1$.
Divide characteristic polynomial by $(u - u_1) = (u + 1)$.
Get other roots from quadratic formula.
One root is double (-1), *hence multiply one of the terms by x.*
Result: $y = c_1\mathrm{e}^{-x} + c_2x\mathrm{e}^{-x} + c_3\mathrm{e}^{5x}$.

e. $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - \frac{\mathrm{d}y}{\mathrm{d}x} - 2y = 3\mathrm{e}^{2x}.$
Particular solution cannot be guessed since RHS is a solution of the complementary homogeneous DE.
Factorise equation to get $(\frac{\partial}{\partial x} + 1)(\frac{\partial}{\partial x} - 2)y = 3\mathrm{e}^{2x}$.
Define $u = (\frac{\partial}{\partial x} - 2)y$ *and solve* $(\frac{\partial}{\partial x} + 1)u = 3\mathrm{e}^{2x}$.
Result for u: $u = \mathrm{e}^{2x} + c_1\mathrm{e}^{-x}$.
Solve $(\frac{\partial}{\partial x} + 1)y = u = \mathrm{e}^{2x} + c_1\mathrm{e}^{-x}$.
Result: $y = x\mathrm{e}^{2x} - \frac{c_1}{3}\mathrm{e}^{-x} + c_2\mathrm{e}^{2x}$.

2. Separating PDEs.
Solve the following homogeneous PDEs by Separation of Variables.

a. $\frac{\partial z}{\partial y} + 2\frac{\partial z}{\partial x} = 0.$
Result: $z(x, y) = c_1\mathrm{e}^{c(2y-x)}$.

b. $\frac{\partial z}{\partial y} + z\frac{\partial z}{\partial x} = 0.$
Result: $z(x, y) = \frac{x+d}{y+e}$.

c. $\frac{\partial z}{\partial y} + x\frac{\partial z}{\partial x} = 0.$
Result: $z(x, y) = d\mathrm{e}^{cy}x^{-c}$.

3. Modelling diffusion.
The diffusion equation in typical physical notation is the inverse of the form we have in the toolbox: In one-dimensional geometry it is: $\frac{\partial c}{\partial t} = \frac{\partial}{\partial x}\left(D\frac{\partial c}{\partial x}\right)$, where D is the diffusion coefficient, which is a constant in well-behaved systems. If this is the case, D can go before the differential: $\frac{\partial c}{\partial t} = D\frac{\partial^2 c}{\partial x^2}$, which is exactly the toolbox case. In less well-behaved systems such as those we're studying in the Materials Physics Group, $D = D(x)$ is really a function of the spatial coordinate. We don't know this dependence explicitly, but the concentration dependence of $D = D(c)$ is known from literature data, and we can guess the concentration profile $c(x)$. What differential equation do we need to solve to get $c(x, t)$? Classify the DE in terms of the properties discussed last week.
Apply the chain rule: $\frac{\partial c}{\partial t} = \frac{\partial}{\partial x}\left(D\frac{\partial c}{\partial x}\right) = \frac{\partial D}{\partial x}\frac{\partial c}{\partial x} + D\frac{\partial^2 c}{\partial x^2}$.
We don't know $\frac{\partial D}{\partial c}$, *but since* $D(c(x))$ *we can apply the chainrule again:* $\ldots = \frac{\partial D}{\partial c}\frac{\partial c}{\partial x}\frac{\partial c}{\partial x} + D\frac{\partial^2 c}{\partial x^2} = \frac{\partial D}{\partial c}\left(\frac{\partial c}{\partial x}\right)^2 + D\frac{\partial^2 c}{\partial x^2}$.

So, this is a non-linear (1st derivative is squared) heterogeneous (LHS contains no x nor any of its derivatives) 2nd order (highest derivative is 2nd order) partial (c is function of x and t) DE.

4. Factorising the characteristic polynomial.
When solving second-order ODEs with constant coefficients, we have reduced the PDE to two ODEs by factorising the characteristic polynomial. The constants in each factor were given as $k_{1,2} = \frac{b}{2} \pm \sqrt{\frac{b^2}{4} - c}$ if the polynomial is $x^2 + bx + c = (x + k_1)(x + k_2)$. Prove this.
$x^2 + bx + c = (x + k_1)(x + k_2) = x^2 + k_1 x + k_2 x + k_1 k_2$.
k_2 is found by quadratic formula: $k_2 = \frac{b}{2} \pm \sqrt{\frac{b^2}{4} - c}$, and $k_1 = k_2 - b = -(\frac{b}{2} \pm \sqrt{\frac{b^2}{4} - c}) = \frac{b}{2} \mp \sqrt{\frac{b^2}{4} - c}$.

Acknowledgement.
Examples 1 are stolen or adapted from *ML Boas; Mathematical Methods in the Physical Sciences, John Wiley, New York (USA) ^{2}1983.*

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