Synchronizing Words and Preparability of States

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Classifying Structures for Operator Algebras and Dynamical Systems
September 16-20, 2013

Abstract: Synchronizing words are a well known topic in the theory of graphs and automata and in symbolic dynamics. They can be reinterpreted as gadgets for the preparation of states on commutative algebras by repeated interactions with another system. This interpretation suggests noncommutative versions which are relevant to the preparation of states in noncommutative algebras and quantum systems.

Joint work with B. Kümmerer, T. Lang, F. Haag.

Road-coloured graphs and synchronizing words

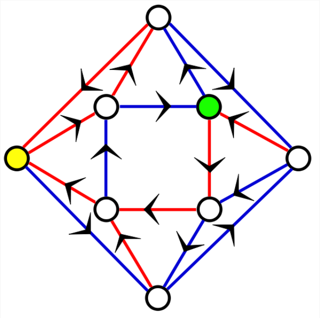

Figure: road-coloured graph, from: Wikimedia Commons

directed (multi-)graph with constant outdegree

road-colouring = bijection between outgoing arrows (for each vertex) and a set of labels or colours [red (r) and blue (b)]

brrbrrbrr leads from any vertex to the yellow vertex

bbrbbrbbr leads from any vertex to the green vertex

These are examples of **synchronizing words**.

Road-colouring problem

Road-colouring problem:

Given a (finite) strongly connected and aperiodic (multi-) graph with constant outdegree, does there always exist a synchronizing word?

Conjectured by Adler and Weiss (1970), complete proof by Trahtman (2009).

Complexity issues:
Finding the shortest synchronizing word is an NP-complete problem (Eppstein 1990).

Computer scientists interpret synchronizing words as reset buttons for finite automata.

Algebraic version: road-colouring

Algebraically:

A set of vertices

C set of colours

A road-colouring graph is nothing but a map

$$\gamma\colon\ A \times C \to A$$

from which we also get the iterated maps

$$\gamma_n\colon\ A \times \prod_1^n C \to A, \quad \gamma_n := \gamma \circ \gamma_{n-1}, \quad \gamma_1 := \gamma$$

In particular this is a
topological Markov chain / subshift of finite type.

Probabilistic interpretation of synchronizing word

Following the instruction given by a synchronizing word involves forgetting the starting point.

Probabilistic version. TFAE:

1. There exists a synchronizing word.
2. For any probability measure μ on A there exists $n \in \mathbb{N}$ and a probability measure ν_n on $\prod_1^n C$ such that for any initial probability measure μ_0 on A

$$\mu = \left(\mu_0 \times \nu_n\right) \circ \gamma_n^{-1}$$

Easy: Prepare the point measure $\mu = \delta_x$ by $\nu_n = \delta_w$ where w is a synchronizing word leading to x. Convex combinations.

Non-commutative topological Markov chain

$\mathcal{A}$, $\mathcal{C}$ $\quad C^*$-algebras

$J : \mathcal{A} \to \mathcal{A} \otimes \mathcal{C}$ $\quad {}^*$-homomorphism

commutative example:
$\mathcal{A} = F(A)$, $\mathcal{C} = F(C)$, $\quad Jf(x,y) = f(\gamma(x,y))$

The corresponding iteration is now

$$J_n \colon \mathcal{A} \to \mathcal{A} \otimes \bigotimes_1^n \mathcal{C}, \quad J_n := \left(J \otimes \mathbb{1}\right) \circ J_{n-1}, \quad J_1 := J$$

We call this a non-commutative topological Markov chain.

Preparability

We say that a state $\rho \in \mathcal{S}(\mathcal{A})$ is **preparable** by $J : \mathcal{A} \to \mathcal{A} \otimes \mathcal{C}$ if there exists a sequence (θ_n) with $\theta_n \in \mathcal{S}(\bigotimes_1^n \mathcal{C})$ such that for any initial state $\rho_0 \in \mathcal{S}(\mathcal{A})$

$$\big(\rho_0 \otimes \theta_n\big) \circ J_n \to \rho \quad \text{if } n \to \infty$$

More natural to consider limits here instead of a finite n.

Choice of topology left open for the moment. But

For commutative algebras from (finite) road-coloured graphs **TFAE**:

1. There is synchronizing word.
2. All states are preparable.

Physical interpretation: Preparing a state (in a quantum system if algebras are non-commutative) by repeated interactions with independent copies of another system.

Attractive feature: The initial state is forgotten.

micromaser experiments

A criterion

Special case: $\mathcal{A} = \mathcal{B}(\mathcal{H})$ and $\mathcal{C}$ a von Neumann algebra.
Stationarity assumption: There exist faithful normal states $\phi \in \mathcal{S}(\mathcal{A})$ and $\psi \in \mathcal{S}(\mathcal{C})$ such that $(\phi \otimes \psi) \circ J = \phi$.

TFAE:

1. All normal states on $\mathcal{A} = \mathcal{B}(\mathcal{H})$ are preparable (weakly or, equivalently, w.r.t. trace norm)
2. The following unital completely positive map Z is ergodic (only trivial fixed points).

$\mathcal{H}_\phi,\ \mathcal{H}_\psi$ GNS-Hilbert spaces w.r.t. $\phi,\ \psi$
$v : \mathcal{H}_\phi \to \mathcal{H}_\phi \otimes \mathcal{H}_\psi$ is the isometry obtained by extending J
Now Z is given by the Stinespring representation

$$Z : \mathcal{B}(\mathcal{H}_\phi) \to \mathcal{B}(\mathcal{H}_\phi), \quad x \mapsto v^* \, x \otimes \mathbb{1} \, v$$

Z is called an **extended transition operator**. Not acting on the original $\mathcal{A} = \mathcal{B}(\mathcal{H})$, in fact $\mathcal{H}_\phi \simeq \mathcal{H} \otimes \mathcal{H}$.

the role of Z

Z^n measures how J_n forgets the $\mathcal{A}$-component of the tensor product:

$$\langle a\Omega_\phi, Z^n(p_\phi)a\Omega_\phi\rangle = \|Q_\phi J_n(a)\|^2_{\psi_n}$$

Ω_ϕ cyclic vector for ϕ, $a \in \mathcal{A} = \mathcal{B}(\mathcal{H})$

p_ϕ the one-dimensional projection onto $\mathbb{C}\Omega_\phi$

Q_ϕ linear extension of the map $a \otimes b \mapsto \phi(a)\, b$

$\|b\|^2_{\psi_n} := \bigotimes_1^n \psi\ (b^* b)$

$$Z \text{ ergodic} \quad \Rightarrow \quad Z^n(p_\phi) \to \mathbb{1} \quad \Rightarrow \quad \|Q_\phi J_n(a)\|_{\psi_n} \to \|a\|_\phi$$

Interpretation: Ergodicity of Z plays a similar role here as the synchronizing word in the easy proof earlier. Indeed, with more work the last property (called asymptotic completeness) makes it possible to prepare arbitrary vector states and then, by convex combinations, arbitrary normal states.