SIMPLICIAL APPROACH
TO THE DE FINETTI THEME

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- a very informal overview
  concentrating on the algebra underlying the probabilities

① $(x_n)_{n\geq 0}$ sequence of random variables (commutative or nc)

$\mathcal{A}$ a (suitable) algebra generated by the x_n

φ a (suitable) state on $\mathcal{A}$

$(x_n)_{n\geq 0}$ is called <u>exchangeable</u> if

$$\varphi(x_{i_1} x_{i_2} \dots x_{i_\ell}) = \varphi(x_{\sigma(i_1)} x_{\sigma(i_2)} \dots x_{\sigma(i_\ell)})$$

for all permutations $\sigma \in S_\infty$

(distribution unchanged by permutations)

$\Leftrightarrow$ there exists a representation $\rho : S_\infty \to \mathrm{Aut}(\mathcal{A}, \varphi)$
such that $\rho(\pi)\, x_n = x_{\pi(n)}$ for all $n \in \mathbb{N}_0$, $\pi \in S_\infty$

notation: $\sigma_k := (k-1\ \ k)$ transposition

$\mathcal{A}_k := \mathcal{A}^{\rho(\sigma_{k+2}, \sigma_{k+3}, \sigma_{k+4}, \dots)}$ fixed point algebra

$\hookrightarrow$ filtration

$\mathcal{A}_{-1} \subset \mathcal{A}_0 \subset \mathcal{A}_1 \subset \dots \subset \mathcal{A}$

$x_k \in \mathcal{A}_k$ (adapted)

<u>constructive procedure</u>: if $\rho: S_\infty \to \mathrm{Aut}(A,\varphi)$ is any rep then we can build exchangeable sequences as follows

- find $x_0 \in A_0$
- then $x_n := \rho(\sigma_n)\, x_{n-1}$ for all $n \geq 1$

proof: check that $\rho(\sigma_n)$ exchanges x_{n-1} and x_n and fixes the other x_k

for example: $\rho(\sigma_n)\, x_{n-2} = \rho(\sigma_n \sigma_{n-2} \dots \sigma_1)\, x_0 = \rho(\sigma_{n-2} \dots \sigma_1)\, \underbrace{\rho(\sigma_n)\, x_0}_{= x_0} = x_{n-2}$ $(n \geq 2)$

$$\rho(\sigma_n)\, x_{n-1} = x_n \,, \quad \rho(\sigma_n)\, x_n = \rho(\sigma_n)^{-1} x_n = x_{n-1}$$

$$\rho(\sigma_n)\, x_{n+1} = \rho(\sigma_n \sigma_{n+1} \sigma_n \sigma_{n-1} \dots \sigma_1)\, x_0 = \rho(\sigma_{n+1} \sigma_n \sigma_{n+1} \sigma_{n-1} \dots \sigma_1)\, x_0 = x_{n+1}$$

② $(x_n)_{n\geq 0}$ is called <u>spreadable</u> if

$$\varphi(x_{i_1} \dots x_{i_\ell}) = \varphi(x_{\sigma(i_1)} \dots x_{\sigma(i_\ell)})$$

only for permutations which are <u>order-preserving</u> (wrt the finitely many variables $x_{i_1}, x_{i_2}, \dots, x_{i_\ell}$ considered)

exchangeable $\Rightarrow$ spreadable

TFAE:

a) $(x_n)_{n\geq 0}$ spreadable

b) There exist $\alpha_0, \alpha_1, \alpha_2, \dots \in \mathrm{End}(A, \varphi)$ such that

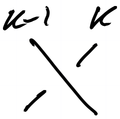
$$\alpha_n : x_k \longrightarrow \begin{cases} x_k & \text{if } k<n \\ x_{k+1} & \text{if } k\geq n \end{cases}$$

(partial shifts)

proof: build arbitrary order-preserving permutations from the α_n, for example

$$x_8 x_5 x_{23} = \alpha_9^{14} \alpha_6^{2} \alpha_0^{5} (x_1 x_0 x_2) \qquad \square$$

how to build spreadable sequences?

observation: if B_∞ is the braid group, $\sigma_k =$ 

Artin generator

and $\rho: B_\infty \to \mathrm{Aut}(A, \varphi)$ (as before, just B_∞ instead of S_∞)

then the constructive procedure

$$x_0 \in A_0$$
$$x_n = \rho(\sigma_n) x_{n-1}$$

still yields a spreadable sequence!

no longer true!

$$\left(\rho(\sigma_n) x_n = \rho(\sigma_n)^{-1} x_n = x_{n-1} \right.$$

but also unnecessary for spreadability)

③ de Finetti's theorem in the form spreadable $\Rightarrow$ conditional i.i.d. over A^{tail}
is proved naturally with partial shifts

- first check that $A^{tail} = A^{\alpha_0}$ (fixed point algebra of shift)
- typical argument in the extremal case $A^{\alpha_0} = \mathbb{C}\cdot\mathbb{1}$

$$\varphi(x_i\, x_j) = \varphi(\alpha_j^k(x_i\, x_j)) = \varphi(x_i\, \alpha_j^k(x_j)) = \varphi(x_i\, \alpha_0^k(x_j))$$

$i<j$

$$= \varphi\Big(x_i \underbrace{\frac{1}{N}\sum_{k=1}^{N} \alpha_0^k(x_j)}_{\to\ \varphi(x_j)\mathbb{1}}\Big) = \varphi(x_i)\,\varphi(x_j)$$

mean ergodic theorem

so the r.v. x_i and x_j are stochastically independent

- This kind of proof even works for nc variables (more complicated, see Köstler JFA 2010)
- In the commutative case we have

exchangeable $\Rightarrow$ spreadable $\Rightarrow$ conditional i.i.d. $\Rightarrow$ exchangeable

so Bas does not give new commutative examples

but for nc variables we find many spreadable sequences which are not exchangeable (Gohm-Köstler CMP 2009)

(4) so what is spreadability algebraically? we take a hint from the properties of the partial shifts,

recall

Δ simplicial category

objects: finite ordered sets $\{0, 1, \dots, n\} = [n]$

morphisms: non-decreasing maps

Δ_S semi-simplicial category

objects: as before

morphisms: (strictly) increasing maps

generated by $\delta^i : [n-1] \to [n]$, $k \to k$ if $k < i$, $k \to k+1$ if $k \geq i$

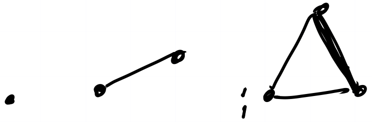
$(i = 0, \dots, n)$ (i left out)

cosimplicial identities: $\delta^j \delta^i = \delta^i \delta^{j-1}$ if $i<j$
(presentation of this category)

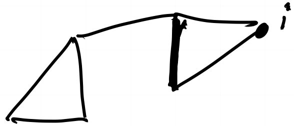
Remark: As relations of a group this would be trivial,
→ Claus' talk tomorrow

We are not talking about a group. But a lot of algebraic structure here.

topology, contravariant functor



simplices



simplicial complexes

$$\to C_2 \to C_1 \to C_0 \to 0$$

for suitable modules C_i

δ_i
face maps

We have a covariant functor (but if we don't look at algebras of random variables but measure spaces and their transformations, it would become contravariant too)

from Homological Algebra:

a <u>semi-cosimplicial object</u> (SCO) in a category $\mathcal{C}$ is a covariant functor F from Δ_S to $\mathcal{C}$, explicitly:

a sequence

$$F[0] \to F[1] \to \dots \to F[n-1] \xrightarrow[(i=0,\dots,n)]{F\delta^i} F[n] \dots$$

A spreadable distribution is the same as a SCO in the category of probability spaces (as algebras of random variables)

$$\begin{array}{c} \\ \\ \end{array} A_0 \subset A_1 \subset \dots \underset{}{\overset{F[n-1]}{\overset{\shortparallel}{A_{n-1}}}} \subset A_n \subset \dots \quad \subset A$$

$$\longrightarrow$$

$$F\delta^i = \alpha_i|_{A_{n-1}}$$

In fact $\alpha_j \alpha_i = \alpha_i \alpha_{j-1}$, $i<j$ (cosimplicial identities).

(Evans + Gohm + Köstler, to appear in RMJ)

(5) What can we learn from that?

under investigation ...

- The braid construction of SCOs seems to be new:
  If B_∞ acts on a set X, $X_n := \{x \in X : \sigma_k x = x \text{ if } k \geq n+2\}$
  then with $F\delta_i = \sigma_{i+1} \cdots \sigma_{n+1} 1_{X_{n-1}} : X_{n-1} \to X_n$ we get a SCO.
- simplicial cohomology: $d = \sum_{i=0}^{n} (-1)^i F\delta^i : F[n-1] \to F[n]$
  for suitable modules
- Yang-Baxter cohomologies?
- Seems to be acyclic if it comes from unitary braid reps (yet unpublished). This applies to the probabilistic settings, to the geometric structure of the α_i as Bernoulli shifts, hence probably related to the de Finetti theorem ...