Kümmerer-Maassen-scattering theory and entanglement

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September 12, 2006

Abstract

We give a new criterion for asymptotic completeness in Kümmerer-Maassen-scattering theory which is more directly in terms of the coupling. To this end we define the semigroup of dual extended transition operators. Asymptotic completeness holds if and only if this semigroup is ergodic. This may also be interpreted in terms of entanglement.

Keywords: scattering, entanglement, transition operator, absorbing state
MSC: Primary 47A40, 47A35; Secondary 47B65, 47L30

1 Coupling representations

In recent years some authors introduced ideas from scattering theory into quantum probability, see the survey article of R. Rebolledo [Re]. More specifically, B. Kümmerer and H. Maassen in [KM] described a scattering theory for Markov chains which is built on the notion of a coupling representation of a Markov process. This is the setting we also adopt here. Our main result (Theorem 4.2) is a new criterion for asymptotic completeness which is more directly formulated in terms of the coupling than the ones in [KM] and which has an interesting interpretation in terms of entanglement of composite systems. The decisive new tool in our approach is the introduction of the semigroup of dual extended transition operators in section 3, which allows us to control the time dependence of the entanglement. This notion may be of independent interest. See also [Go1, Go2] for some further background material to our approach.

Let us quickly recall the setting. For more details and motivations the reader should consult [KM]. We give quotations for the main definitions and introduce some small modifications. In particular we restrict to a one-sided version with respect to time, which is all what is needed in this paper.

A (noncommutative) probability space is a pair $(\mathcal{A}, \phi)$ consisting of a von Neumann algebra $\mathcal{A}$ and a faithful normal state ϕ. By $T : (\mathcal{A}, \phi) \to (\mathcal{B}, \psi)$ we denote a normal unital completely positive map $T : \mathcal{A} \to \mathcal{B}$ with $\psi \circ T = \phi$. A (noncommutative) stochastic process is a family of normal $^*-$homomorphisms ('random variables') $j_t : (\mathcal{A}, \phi) \to (\hat{\mathcal{A}}, \hat{\phi})$, where $t \in \mathbb{T}_+$ with $\mathbb{T}_+ = \mathbb{N}_0$ or $\mathbb{T}_+ = \{0 \leq t \in \mathbb{R}\}$. Writing $i = j_0$ the process is called stationary if there is a semigroup of normal $^*-$homomorphisms $\alpha_t : (\hat{\mathcal{A}}, \hat{\phi}) \to (\hat{\mathcal{A}}, \hat{\phi})$ such that $j_t = \alpha_t \circ i$ for all t. If $\mathbb{T}_+ = \{0 \leq t \in \mathbb{R}\}$ the semigroup is assumed to be pointwise weak$^*-$continuous.

A (noncommutative) white noise is a (noncommutative) probability space $(\hat{\mathcal{C}}, \hat{\psi})$ with a semigroup of normal $^*-$homomorphisms $\sigma_t : (\hat{\mathcal{C}}, \hat{\psi}) \to (\hat{\mathcal{C}}, \hat{\psi})$ (where $t \in \mathbb{T}_+$, pointwise weak$^*-$continuous for real parameters) and a filtration $\{\mathcal{C}_{[s,t)},\ 0 \leq s < t \leq \infty\}$ of von Neumann subalgebras compatible with $\{\sigma_t\}$ in the sense that $\sigma_t(\mathcal{C}_{[u,v)}) = \mathcal{C}_{[u+t,v+t)}$ and such that

- $\hat{\mathcal{C}} = \mathcal{C}_{[0,\infty)}$
- $\mathcal{C}_{[s,t)}$ is generated by $\mathcal{C}_{[s,u)}$ and $\mathcal{C}_{[u,t)}$ if $0 \leq s < u < t$
- $\mathcal{C}_{[s,t)}$ and $\mathcal{C}_{[u,v)}$ are independent subalgebras of $(\hat{\mathcal{C}}, \hat{\psi})$ whenever the intervals $[s,t)$ and $[u,v)$ are disjoint.

Independence is meant in the sense introduced by B. Kümmerer which allows to treat many different notions of noncommutative independence simultaneously, including commutative schemes, CCR, CAR, free and q-white noises: Two subalgebras $\mathcal{B}_1, \mathcal{B}_2$ in $(\mathcal{B}, \psi)$ are called independent if $\psi(x_1 x_2) = \psi(x_1)\psi(x_2)$ for all $x_1 \in \mathcal{B}_1$ and $x_2 \in \mathcal{B}_2$ and if there are conditional expectations P_i from $\mathcal{B}$ onto $\mathcal{B}_i$ $(i = 1, 2)$ which preserve the state ψ. Using the module property of conditional expectations one concludes that this relation of independence is symmetric and that

$$\psi(xzy) = \psi(xy)\, \psi(z)$$

if $x, y \in \mathcal{B}_1, z \in \mathcal{B}_2$ and $\mathcal{B}_1, \mathcal{B}_2$ independent. But higher correlations are not determined in general.

A stationary process $(\hat{\mathcal{A}}, \hat{\phi}, \alpha_t, i)$ has a coupling representation of tensor type if $\hat{\mathcal{A}} = i(\mathcal{A}) \otimes \hat{\mathcal{C}}$ with $\hat{\phi} = \phi \otimes \hat{\psi}$ and $\mathbb{1} \otimes \hat{\mathcal{C}}$ is invariant for the α_t and $(\hat{\mathcal{C}}, \hat{\psi})$ with σ_t given by $\mathbb{1} \otimes \sigma_t(c) := \alpha_t(\mathbb{1} \otimes c)$ is a white noise and $\alpha_t\, i(\mathcal{A}) \subset i(\mathcal{A}) \otimes \mathcal{C}_{[0,t)}$ for all $t \in \mathbb{T}_+$. Here we always identify $i(\mathcal{A})$ and $i(\mathcal{A}) \otimes \mathbb{1}$. Given such a coupling representation it is possible to think of the random variables j_t of the process as

$$j_t : \mathcal{A} \to i(\mathcal{A}) \otimes \mathcal{C}_{[0,t)} \subset \hat{\mathcal{A}}$$

and we have for $a \in \mathcal{A}$, $c \in \hat{\mathcal{C}}$

$$\alpha_t(i(\mathcal{A}) \otimes c) = j_t(a)\, (\mathbb{1} \otimes \sigma_t(c))$$

This decomposition of the time evolution α_t into a coupling given by j_t and a white noise evolution σ_t is fundamental for the things to come. It can be checked that a coupling representation implies a noncommutative Markov property for the process. In particular, introducing the tensor type conditional expectation $P : (\hat{\mathcal{A}}, \hat{\phi}) \to (\mathcal{A}, \phi)$ defined by evaluating the state $\hat{\psi}$ on $\hat{\mathcal{C}}$, we get with $T_t := P\, j_t : (\mathcal{A}, \phi) \to (\mathcal{A}, \phi)$ a semigroup of transition operators for this Markov process.

A two-sided version with parameters in $\mathbb{Z}$ or $\mathbb{R}$ may be recovered by enlarging the algebras using stationarity. In [KM] the authors proceed to discuss scattering theory for the two-sided version, considering the evolution of the Markov process as a perturbation of the white noise evolution. They define asymptotic completeness in the usual way and get the following criterion which can be formulated using only the one-sided version introduced above. We shall use this criterion later while there is no need for us here to discuss the scattering explicitly.

Criterion: ([KM], Theorem 3.3)
A coupling representation is asymptotically complete if and only if

$$\lim_{t\to\infty} \|Q\, \alpha_t\, i(a)\|_{\hat{\psi}} = \|a\|_\phi$$

for all $a \in \mathcal{A}$. Here $Q : (\hat{\mathcal{A}}, \hat{\phi}) \to (\hat{\mathcal{C}}, \hat{\psi})$ denotes the tensor type conditional expectation defined by evaluating the state ϕ on $i(\mathcal{A})$. Further $\|\cdot\|_\phi$ is the norm induced by the inner product $(x, y) \mapsto \phi(x^* y)$, similar for other states. With respect to these norms the conditional expectations become orthogonal projections.

Intuitively the criterion says that after a long time the contents of $\mathcal{A}$ are gone into the noise algebra $\hat{\mathcal{C}}$. This is sometimes not so easy to check because it involves the full dynamics α_t. We do not consider the further development in [KM] about this but construct a new criterion. In the following section we prepare it by some considerations about entanglement.

2 Entanglement for noncommutative probability spaces

The most elementary setting for discussing entanglement in quantum mechanics consists in considering a composite quantum system described by a tensor product of Hilbert spaces $\mathcal{H}_1 \otimes \mathcal{H}_2$ and a pure state described by a unit vector $\chi \in \mathcal{H}_1 \otimes \mathcal{H}_2$. This state is called entangled if χ cannot be written as an elementary tensor $\chi_1 \otimes \chi_2$. See [NC] for a detailed account of different aspects of entanglement.

Entanglement can be checked by computing reduced density operators. We use Dirac notation $|\chi\rangle\langle\chi|$ for the one-dimensional projection onto $\mathbb{C}\chi$ and write Tr_2 for the partial trace evaluated in $\mathcal{H}_2$. Such a notation will also be used later in similar cases. Then the reduced density operator is defined as

$$\chi^{\mathcal{H}_1} := Tr_2(|\chi\rangle\langle\chi|),$$

which is a positive trace class operator on $\mathcal{H}_1$ with trace equal to 1. Such operators are called density operators. The state given by χ is entangled if and only if the rank of the associated reduced density operator is strictly larger than 1. A more precise quantitative measure of entanglement can be given by using von Neumann entropy which for a density operator ρ is defined by $h(\rho) := -Tr(\rho \log \rho) \geq 0$. Then the state given by χ is entangled if and only if $h(\chi^{\mathcal{H}_1}) > 0$, and decrease of entropy may be interpreted as decrease of entanglement.

If $\{\delta_i\}$ is an ONB of $\mathcal{H}_1$ and $\chi = \sum_i \delta_i \otimes \xi_i$, then with respect to this basis the density operator becomes a density matrix with entries $\left(\chi^{\mathcal{H}_1}\right)_{jk} = \langle \xi_k, \xi_j \rangle$. This is a Gram matrix, and if the $\{\xi_i\}$ are random variables in some sense, as will be the case later, we may think of it as a kind of covariance matrix describing the entanglement in question.

We now define a version of this formalism whose physical interpretation is not quite clear but whose mathematical usefulness will appear soon when we apply it to the scattering problem. Let two noncommutative probability spaces $(\mathcal{B}_1, \psi_1)$ and $(\mathcal{B}_2, \psi_2)$ be given. Applying the GNS-construction to $\mathcal{B}_1 \otimes \mathcal{B}_2$ with the state $\psi_1 \otimes \psi_2$ we get a Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$, a representation π of $\mathcal{B}_1 \otimes \mathcal{B}_2$ on $\mathcal{H}_1 \otimes \mathcal{H}_2$ and a unit vector $\Omega = \Omega_1 \otimes \Omega_2 \in \mathcal{H}_1 \otimes \mathcal{H}_2$ such that $\psi_1 \otimes \psi_2(b_1 \otimes b_2) = \langle \Omega, \pi(b_1 \otimes b_2)\Omega \rangle$. For $b \in \mathcal{B}_1 \otimes \mathcal{B}_2$ with $\|\pi(b)\Omega\| = \|b\|_{\psi_1 \otimes \psi_2} = 1$ we can define

$$b^{\mathcal{H}_1} := (\pi(b)\Omega)^{\mathcal{H}_1} = Tr_2(|\pi(b)\Omega\rangle\langle\pi(b)\Omega|),$$

and we call this the covariance operator of b (with respect to $(\mathcal{B}_1, \psi_1) \otimes (\mathcal{B}_2, \psi_2)$). It describes a kind of entanglement for observables.

3 The dual extended transition operators

Let $j : (\mathcal{A}, \phi) \to (\hat{\mathcal{A}}, \hat{\phi})$ be a $^*-$homomorphism ('random variable'). Then we have $(\mathcal{H}, \Omega)$ and $(\hat{\mathcal{H}}, \hat{\Omega})$ as GNS-Hilbert spaces with cyclic vectors. We suppress the notation for the representations and write $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ and $\hat{\mathcal{A}} \subset \mathcal{B}(\hat{\mathcal{H}})$ instead. Further we have a map

$$v : \mathcal{H} \to \hat{\mathcal{H}}, \quad a\Omega \mapsto j(a)\hat{\Omega} \quad (a \in \mathcal{A}).$$

We can check that v is an isometry:

$$\langle va_1\Omega, va_2\Omega \rangle = \langle j(a_1)\hat{\Omega}, j(a_2)\hat{\Omega} \rangle = \langle \hat{\Omega}, j(a_1^* a_2)\hat{\Omega} \rangle$$

$$= \hat{\phi}(j(a_1^* a_2)) = \phi(a_1^* a_2) = \langle a_1\Omega, a_2\Omega \rangle$$

We now return to the setting of section 1, i.e. we consider a stationary Markov process in a coupling representation of tensor type which we want to examine for asymptotic completeness. The random variables

$$j_t : \mathcal{A} \to i(\mathcal{A}) \otimes \mathcal{C}_{[0,t)} \subset \hat{\mathcal{A}}$$

of the process give rise to isometries

$$v_t : \mathcal{H} \to \mathcal{H} \otimes \mathcal{K}_{[0,t)} \subset \hat{\mathcal{H}},$$

where the Hilbert spaces are the GNS-spaces corresponding to the algebras above. For the cyclic vectors we write $\Omega \in \mathcal{H}$, $\Omega_{[0,t)} \in \mathcal{K}_{[0,t)}$ and $\hat{\Omega} \in \hat{\mathcal{H}}$. Note also that we identify $a\Omega$ and $i(a)\Omega$. We have $\hat{\Omega} = \Omega \otimes \Omega_{[0,\infty)}$, and embeddings are given in a natural way by

$$\mathcal{H} \simeq \mathcal{H} \otimes \Omega_{[0,\infty)} \subset \mathcal{H} \otimes \mathcal{K}_{[0,\infty)} \simeq \hat{\mathcal{H}},$$

$$\mathcal{H} \otimes \mathcal{K}_{[0,t)} \simeq \mathcal{H} \otimes \mathcal{K}_{[0,t)} \otimes \Omega_{[t,\infty]} \subset \hat{\mathcal{H}}, \quad etc.$$

Definition 3.1 *The normal unital completely positive map*

$$Z_t' : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}), \quad x \mapsto v_t^* \, x \otimes \mathbb{1} \, v_t$$

is called the dual extended transition operator corresponding to j_t.

Some remarks: The operator Z_t' should not be confused with the operator Z_t appearing in the paper [KM] which acts on different spaces. The reason for the notation $'$ and for the terminology is given by the fact that Z_t' is an extension of the dual map T_t' of the transition operator T_t of the Markov process, see [Go1, Go2] for more details. We shall not need this fact here but work directly with the definition above.

In a similar way we can also associate isometries to the full time evolution α_t of the Markov process:

$$\hat{v}_t : \hat{\mathcal{H}} \to \hat{\mathcal{H}}, \quad \hat{a}\hat{\Omega} \mapsto \alpha_t(\hat{a})\hat{\Omega} \quad (\hat{a} \in \hat{\mathcal{A}}).$$

Let us write $< \cdot, \cdot >$ for the duality between trace class operators $\mathcal{T}(\cdot)$ and bounded operators $\mathcal{B}(\cdot)$. Further $Tr_{[0,t)}$ denotes the partial trace evaluated on $\mathcal{K}_{[0,t)}$.

Lemma 3.2 *For all $\hat{\rho} \in \mathcal{T}(\hat{\mathcal{H}})$ and $x \in \mathcal{B}(\mathcal{H})$ we get*

$$< Tr_{[0,\infty)}(\hat{v}_t \, \hat{\rho} \, \hat{v}_t^*), x > = < Tr_{[0,\infty)}(\hat{\rho}), Z_t'(x) > .$$

Proof: It suffices to consider

$$\begin{aligned} \hat{\rho} = \rho \otimes \rho_{[0,\infty)} &= |i(a)\Omega\rangle\langle i(b)\Omega| \otimes |c\,\Omega_{[0,\infty)}\rangle\langle d\,\Omega_{[0,\infty)}| \\ &= |i(a) \otimes c\,\hat{\Omega}\rangle \; \langle i(b) \otimes d\,\hat{\Omega}| \end{aligned}$$

with $a, b \in \mathcal{A}$ and $c, d \in \mathcal{C}_{[0,\infty)}$. We can write $j_t(a) = \sum_n a_n \otimes a_n^{(t)}$ with $a_n \in \mathcal{A}$ and $a_n^{(t)} \in \mathcal{C}_{[0,t)}$ and $v_t\, a\,\Omega = \sum_n \delta_n \otimes a_n^{(t)} \Omega_{[0,t)}$ with $\delta_n = a_n \Omega \in \mathcal{H}$. Similarly $v_t\, b\,\Omega = \sum_m \delta_m \otimes b_m^{(t)} \Omega_{[0,t)}$ with $b_m^{(t)} \in \mathcal{C}_{[0,t)}$. If $\mathcal{H}$ is infinite dimensional then the sums are suitable limits. We note that

$$Tr_{[0,t)}(v_t |a\Omega\rangle\langle b\Omega| v_t^*) = Tr_{[0,t)} \left(|\sum_n \delta_n \otimes a_n^{(t)} \Omega_{[0,t)}\rangle \langle \sum_m \delta_m \otimes b_m^{(t)} \Omega_{[0,t)}| \right)$$

$$= \sum_{n,m} |\delta_n\rangle\langle\delta_m| \ \langle b_m^{(t)} \Omega_{[0,t)}, a_n^{(t)} \Omega_{[0,t)} \rangle = \sum_{n,m} |\delta_n\rangle\langle\delta_m| \ \hat{\psi} \left((b_m^{(t)})^* a_n^{(t)} \right).$$

Then we compute

$$Tr_{[0,\infty)}(\hat{v}_t\, \hat{\rho}\, \hat{v}_t^*) = Tr_{[0,\infty)} \left(|j_t(a)\, (\mathbb{1} \otimes \sigma_t(c))\ \hat{\Omega}\rangle\langle j_t(b)\, (\mathbb{1} \otimes \sigma_t(d))\ \hat{\Omega}| \right)$$

$$= Tr_{[0,\infty)} \left(|\sum_n a_n \otimes a_n^{(t)} \sigma_t(c)\, \hat{\Omega}\rangle \langle \sum_m a_m \otimes b_m^{(t)} \sigma_t(d)\, \hat{\Omega}| \right)$$

$$= \sum_{n,m} |\delta_n\rangle\langle\delta_m| \ \langle b_m^{(t)} \sigma_t(d) \Omega_{[0,\infty)}, a_n^{(t)} \sigma_t(c) \Omega_{[0,\infty)} \rangle$$

$$= \sum_{n,m} |\delta_n\rangle\langle\delta_m| \ \hat{\psi} \left(\sigma_t(d^*) (b_m^{(t)})^* a_n^{(t)} \sigma_t(c) \right)$$

and using the independence property of white noise and our previous calculation we can continue with

$$= \sum_{n,m} |\delta_n\rangle\langle\delta_m| \ \hat{\psi} \left((b_m^{(t)})^* a_n^{(t)} \right) \hat{\psi}(\sigma_t(d^* c)) \ = \ Tr_{[0,t)}(v_t |a\Omega\rangle\langle b\Omega| v_t^*)\ Tr(\rho_{[0,\infty)}).$$

Let us write $(Z'_t)_*$ for the preadjoint of Z'_t with respect to the duality between $\mathcal{T}(\mathcal{H})$ and $\mathcal{B}(\mathcal{H})$. Explicitly:

$$(Z'_t)_* : \mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{H}), \quad \rho \mapsto Tr_{[0,t)}(v_t\, \rho\, v_t^*).$$

Then we find for $x \in \mathcal{B}(\mathcal{H})$

$$< Tr_{[0,\infty)}(\hat{v}_t\, \hat{\rho}\, \hat{v}_t^*), x > \quad = \quad < Tr_{[0,t)}(v_t\, \rho\, v_t^*), x > Tr(\rho_{[0,\infty)})$$

$$= < (Z'_t)_*(\rho), x > Tr(\rho_{[0,\infty)}) \ = < \rho, Z'_t(x) > Tr(\rho_{[0,\infty)}) \ = < Tr_{[0,\infty)}(\hat{\rho}), Z'_t(x) > .$$

The lemma is proved.

Theorem 3.3 *$(Z'_t)_*(\hat{a}^{\mathcal{H}}) = \alpha_t(\hat{a})^{\mathcal{H}}$ for all $\hat{a} \in \hat{\mathcal{A}}$.*

The notations $\hat{a}^{\mathcal{H}}$ and $\alpha_t(\hat{a})^{\mathcal{H}}$ refer to the covariance operator introduced in section 2, with respect to $(i(\mathcal{A}), \phi) \otimes (\hat{\mathcal{C}}, \hat{\psi})$. Further $(Z'_t)_*$ is the preadjoint of Z'_t, as in the proof of Lemma 3.2 above. Thus the theorem tells us that the dual extended transition operators describe the time evolution of the covariance operators with respect to the decomposition into the system $i(\mathcal{A})$ and the white noise $\hat{\mathcal{C}}$ if the Markovian time evolution is applied to the observables.

Proof: It suffices to insert $\hat{\rho} = |\hat{a}\hat{\Omega}\rangle\langle\hat{a}\hat{\Omega}|$ into Lemma 3.2. We get

$$< \alpha_t(\hat{a})^{\mathcal{H}}, x > \;=\; < Tr_{[0,\infty)}(|\alpha_t(\hat{a})\hat{\Omega}\rangle\langle\alpha_t(\hat{a})\hat{\Omega}|), x >$$

$$= < Tr_{[0,\infty)}(\hat{v}_t\, \hat{\rho}\, \hat{v}_t^*), x > = < Tr_{[0,\infty)}(\hat{\rho}), Z'_t(x) > = < \hat{a}^{\mathcal{H}}, Z'_t(x) > .$$

Proposition 3.4 *$\{Z'_t\}_{t \in \mathbb{T}_+}$ is a semigroup (with $Z'_0 = \mathbb{1}$ and pointwise weak*-continuous for real parameters). The state $\langle \Omega, \cdot\, \Omega\rangle$ is invariant.*

Proof: $Z'_0 = \mathbb{1}$ is immediate, and the continuity follows from the corresponding assumption about the Markovian time evolution. Because elements of the form $\hat{a}^{\mathcal{H}}$ span $\mathcal{T}(\mathcal{H})$ it suffices for the proof of the semigroup property to consider $< \hat{a}^{\mathcal{H}}, Z'_t(x) >$. We compute

$$< \hat{a}^{\mathcal{H}}, Z'_{t+s}(x) > = < \alpha_{t+s}(\hat{a})^{\mathcal{H}}, x > = < \alpha_s(\alpha_t(\hat{a}))^{\mathcal{H}}, x >$$

$$= < \alpha_t(\hat{a})^{\mathcal{H}}, Z'_s(x) > \;=\; < \hat{a}^{\mathcal{H}}, Z'_t\, Z'_s(x) > .$$

This implies $Z'_{t+s} = Z'_t\, Z'_s$. The invariance of the state $\langle \Omega, \cdot\, \Omega\rangle$ follows from $v_t\Omega = \Omega \otimes \Omega_{[0,t)}$.

4 A criterion for asymptotic completeness

To apply the results of section 3 to scattering we need some results about ergodic properties of positive maps. While these are certainly well known we have not found a detailed reference and thus include the following

Lemma 4.1 *Let $\{S_t\}_{t \in \mathbb{T}_+}$ be a semigroup of normal positive unital maps on $\mathcal{B}(\mathcal{H})$ (with $S_0 = \mathbb{1}$). Further let $\Omega \in \mathcal{H}$ be a unit vector and $p := |\Omega\rangle\langle\Omega|$. Then the following assertions are equivalent:*

(a) $\{S_t\}_{t \in \mathbb{T}_+}$ is ergodic, i.e. for $t > 0$ the only fixed points of S_t are $\mathbb{C}\mathbb{1}$, and the state $\langle \Omega, \cdot\, \Omega\rangle$ is invariant.

(b) The state $\langle \Omega, \cdot\, \Omega\rangle$ is absorbing for $\{S_t\}_{t \in \mathbb{T}_+}$. i.e.

$$\lim_{t \to \infty} \langle \rho, S_t(x)\rangle = \langle \Omega, x\Omega\rangle$$

for all density operators ρ in $\mathcal{T}(\mathcal{H})$ and $x \in \mathcal{B}(\mathcal{H})$.

(c) $\lim_{t\to\infty} S_t(p) = 1\!\!\mathrm{I}$ *(strong operator limit or, equivalently, weak*-limit)*

(d) $\lim_{t\to\infty} < (S_t)_*(|\xi\rangle\langle\xi|), p >= 1$ *for a dense set of unit vectors* $\xi \in \mathcal{H}$.

(e) $\lim_{t\to\infty} \|(S_t)_*(\rho) - p\|_1 = 0$ *for all density operators* ρ.
Here $\|\cdot\|_1$ *denotes the trace class norm of* $\mathcal{T}(\mathcal{H})$.

Proof: $(b) \Rightarrow (a)$. Clearly an absorbing state is invariant. Let $x \in \mathcal{B}(\mathcal{H})$ be a fixed point of S_r for some $r > 0$. If $x \notin \mathbb{C}1\!\!\mathrm{I}$ then there exist density operators ρ_1, ρ_2 such that $< \rho_1, x > \neq < \rho_2, x >$. But then we also have $\lim_{n\to\infty} < \rho_1, S_{nr}(x) > \neq \lim_{n\to\infty} < \rho_2, S_{nr}(x) >$.
$(a) \Rightarrow (c)$. From invariance we conclude that $S_t(p) \geq p$ and thus there exists $q := \lim_{t\to\infty} S_t(p)$ as a strong operator limit. Because q is a fixed point and $p \leq q \leq 1\!\!\mathrm{I}$, ergodicity implies $q = 1\!\!\mathrm{I}$.
$(c) \Rightarrow (b)$. Decompose $x \in \mathcal{B}(\mathcal{H})$ as $x = pxp + (1\!\!\mathrm{I} - p)xp + px(1\!\!\mathrm{I} - p) + (1\!\!\mathrm{I} - p)x(1\!\!\mathrm{I} - p)$. Because $pxp =< p, x > p$ we get $\lim_{t\to\infty} S_t(pxp) =< p, x > 1\!\!\mathrm{I} = \langle \Omega, x\Omega\rangle 1\!\!\mathrm{I}$. The other terms vanish for $t \to \infty$, as can be seen by applying the Cauchy-Schwarz inequality for the states $y \mapsto \langle \Omega, S_t(y)\Omega\rangle$. Note that weak*-convergence in (c) is enough to get (b) in this way.
We have shown $(a) \Leftrightarrow (b) \Leftrightarrow (c)$. Then $(c) \Leftrightarrow (d)$ follows from duality and from the fact that one-dimensional projections span $\mathcal{T}(\mathcal{H})$. $(e) \Rightarrow (d)$ is immediate and the converse direction $(d) \Rightarrow (e)$ is shown in [Ta], III.5.11.

Theorem 4.2 *Let* $\{Z'_t\}_{t\in\mathbb{T}_+}$ *be the semigroup of dual extended transition operators. The following assertions are equivalent:*

(1) The coupling representation is asymptotically complete.

(2) $\{Z'_t\}_{t\in\mathbb{T}_+}$ *satsfies the conditions of Lemma 4.1 (with* Ω *the cyclic vector corresponding to* ϕ*).*

Proof: If $a \in \mathcal{A}$ with $\|a\|_\phi = 1$, then using the same notation as in the proof of Lemma 3.2 we can write

$$v_t a\Omega = \alpha_t\, i(a)\,\Omega \otimes \Omega_{[0,t)} = \sum_n \delta_n \otimes a_n^{(t)}\Omega_{[0,t)}$$

with $\{\delta_n\}$ an ONB of $\mathcal{H}$. We can choose $\delta_0 = \Omega$ and get $a_0^{(t)} = Q\alpha_t i(a)$ with Q the conditional expectation from $(\hat{\mathcal{A}}, \hat{\phi})$ onto $(\hat{\mathcal{C}}, \hat{\psi})$. Using the criterion for asymptotic completeness mentioned in section 1 we see that asymptotic completeness is equivalent to

$$\lim_{t\to\infty} \|a_0^{(t)}\|_{\hat{\psi}} = \|a\|_\phi = 1 \quad \textit{for all } a \in \mathcal{A} \textit{ with } \|a\|_\phi = 1.$$

On the other hand, we can use Theorem 3.3 to conclude that

$$(Z'_t)_*(|a\Omega\rangle\langle a\Omega|) = (Z'_t)_*(i(a)^{\mathcal{H}}) = (\alpha_t\, i(a))^{\mathcal{H}},$$

which has matrix entries

$$\langle \delta_n, (Z'_t)_*(|a\Omega\rangle\langle a\Omega|)\,\delta_m\rangle = \langle a_m^{(t)}\Omega_{[0,t)}, a_n^{(t)}\Omega_{[0,t)}\rangle,$$

in particular

$$\langle \Omega, (Z'_t)_*(|a\Omega\rangle\langle a\Omega|)\,\Omega\rangle = \|a_0^{(t)}\Omega_{[0,t)}\|^2 = \|a_0^{(t)}\|^2_{\hat{\psi}}.$$

Now we see that $\lim_{t\to\infty}\|a_0^{(t)}\|_{\hat{\psi}} = \|a\|_\phi = 1$ for all $a \in \mathcal{A}$ with $\|a\|_\phi = 1$ corresponds to condition (d) of Lemma 4.1 for the semigroup $\{Z'_t\}_{t\in\mathbb{T}_+}$. Theorem 4.2 is proved.
Because $\{Z'_t\}_{t\in\mathbb{T}_+}$ describes the time evolution of entanglement, see Theorem 3.3, we can also interpret Theorem 4.2 in such terms: Asymptotic completeness corresponds to a decay of entanglement between the system $i(\mathcal{A})$ and the noise $\hat{\mathcal{C}}$. A numerical version of this can be given by using entropy.

Corollary 4.3 *The coupling representation is asymptotically complete if and only if the von Neumann entropies of all covariance operators w.r.t.* $(i(\mathcal{A}),\phi)\otimes(\hat{\mathcal{C}},\hat{\psi})$ *tend to* 0 *for* $t\to\infty$.

Proof: In terms of eigenvalue lists both the convergence in Theorem 4.2(2) and the asymptotic vanishing of the entropies mean that for large t the largest eigenvalue of a covariance operator has multiplicity 1 and tends to 1, while all other eigenvalues tend to 0.

The most elementary example is the following: Choose $\mathbb{T}_+ = \mathbb{N}_0$, $\mathcal{A} = \mathcal{C} = \mathbb{C}^2$ (commutative algebra with canonical basis $\delta_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\delta_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$), $\phi = \psi : \mathbb{C}^2 \to \mathbb{C}$ with $\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \mapsto \frac{1}{2}(\lambda_1+\lambda_2)$, $\hat{\mathcal{C}} =$ weak closure of $\bigotimes_{n=0}^\infty \mathcal{C}$ with respect to the state $\hat{\psi} = \bigotimes_{n=0}^\infty \psi$, and σ the right shift on $\hat{\mathcal{C}}$. If further the coupling $j_1 : \mathcal{A} \to \mathcal{A}\otimes\mathcal{C}$ is chosen as

$$j_1(\lambda_1\delta_1 + \lambda_2\delta_2) = \lambda_1(\mathbb{1}\otimes\delta_1) + \lambda_2(\mathbb{1}\otimes\delta_2),$$

then a short computation yields that for $x \in \mathcal{B}(\mathbb{C}^2)$

$$Z'_1(x) = \langle \Omega, x\Omega\rangle\,\mathbb{1}.$$

Then $Z'_n = (Z'_1)^n = Z'_1$ and $\langle \Omega, \cdot\,\Omega\rangle$ is absorbing. Thus we have asymptotic completeness in this case. On the other hand, if we choose

$$j_1(\lambda_1\delta_1 + \lambda_2\delta_2) = \lambda_1(\delta_1\otimes\delta_1 + \delta_2\otimes\delta_2) + \lambda_2(\delta_1\otimes\delta_2 + \delta_2\otimes\delta_1),$$

then we get $Z'_1 : \mathcal{B}(\mathbb{C}^2) \to \mathcal{B}(\mathbb{C}^2)$ with

$$x = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \mapsto \frac{1}{2}\begin{pmatrix} x_{11}+x_{22} & x_{12}+x_{21} \\ x_{12}+x_{21} & x_{11}+x_{22} \end{pmatrix}.$$

In this case every $2\times 2-$matrix of the form $\begin{pmatrix} a & b \\ b & a \end{pmatrix}$ is a fixed point and we conclude that the coupling representation is not asymptotically complete. Note that in both cases the transition operator $T : \mathcal{A} \to \mathcal{A}$ of the Markov process is given by the stochastic matrix $\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. This shows that the transition operator does not determine whether the coupling representation is asymptotically complete or not. As shown in Theorem 4.2, the dual extended transition operators encode more information about the coupling and suffice to perform this task.

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