A Spectral Classification of Operators related to Polynomial Boundedness *

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Abstract

A local version of the concept of polynomial boundedness for operators on Banach spaces is defined and its relations to functional calculi are examined. For certain positive operators on L^∞–spaces, especially for endomorphisms, lack of local polynomial boundedness corresponds to mixing properties. In particular, we give a new characterization of the weak mixing property. Some results extend to more general C^*–algebras. This is done by constructing certain topological embeddings of the unit vector base of $l^1(\mathbb{N}_0)$ into the orbits of an operator. To analyze the underlying structure we introduce the concept of a transition set. We compute transition sets for the shift operator on $l^1(\mathbf{Z})$ and show how to define a corresponding similarity invariant.

Contents:

1 Introduction

Reading the classical monograph "Harmonic Analysis of Operators in Hilbert Spaces" by B. Sz.-Nagy and C. Foias [15] provides a strong impression of the usefulness of large functional calculi in establishing a detailed structure theory of operators. The property of polynomial boundedness is a necessary condition for the possibility of defining these calculi. On the other hand, a rather different sort of behaviour should be expected if we consider operators which are not polynomially bounded.

In the second section we develop the concept of local polynomial boundedness. With the help of this point of view it is possible to obtain spectral properties of an operator which are equivalent to the existence of certain functional calculi. Results which are implicit in the work of many authors

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using functional calculi can be summarized in a convenient way. Conversely we may characterize those operators on Banach spaces, which do not allow an approach by large functional calculi, by their lack of local polynomial boundedness. In fact, the main idea behind our work in the following sections has been to consider degrees of deviation from local polynomial boundedness and to use the classification evolving in this manner as a new structure theory of operators.

In the third section we show that for endomorphisms and some other positive operators on L^∞–spaces this classification is closely related to the classification concerning mixing properties, known from ergodic theory. Some of the results extend to more general C^*–algebras. The classical Rohlin lemma and its modern version by A.Connes are used, and we give an equivalent characterization of the weak mixing property in terms of local polynomial boundedness, using a result of H. Furstenberg. The relation between ergodic theory and polynomial boundedness is established by certain topological embeddings of $l^1(\mathbb{N}_0)$ into the orbits of the operator. A lemma by H. P. Rosenthal and L. Dor becomes relevant for operator theory as a tool to identify such embeddings. For shift operators these constructions are particularly easy.

Indeed the existence of such embeddings is completely opposite to any polynomial boundedness properties. We try to encode the information obtained in this way concerning deviation from polynomial boundedness into the socalled transition set. The fourth section is devoted to this concept.

For the shift operator on $l^1(\mathbf{Z})$ all transition sets can be computed explicitly from symmetry properties of the set of zeros of Fourier transforms. Extending the definition of the transition set, we can define an invariant for similarity of operators. For example, if we find nonempty transition sets for an operator, then infinitely many cyclic subspaces can be constructed such that its restrictions are not similar. The complexity of a structure theory for operators of this class is already evident from that.

2 Local Functional Calculi

First of all we fix some notation which will be used throughout the text. The open unit disk $\{z \in \mathbb{C} : |z| < 1\}$ is denoted by D, and for its boundary ∂D we use a parametrization by $[0, 2\pi) \ni t \mapsto e^{it} \in \partial D$.
dt denotes Lebesgue measure on $[0, 2\pi)$, and $\mathcal{M}[0, 2\pi)$ is the set of all (complex, regular, finite) Borel measures on $[0, 2\pi)$. For any suitable function f on ∂D we get Fourier coefficients $\hat{f}(n) := \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-int}\, dt$.
We consider the following function algebras on ∂D:
$L^\infty(\partial D)$, the algebra of dt-essentially bounded functions on ∂D,
$C(\partial D)$, the algebra of continuous functions on ∂D,
$A(\partial D) := \left\{ f \in C(\partial D) : \|f\|_A := \sum_{n \in \mathbf{Z}} |\hat{f}(n)| < \infty \right\}$, the Wiener–algebra, and in each case the subalgebra of those functions whose Fourier coefficients with negative indizes all vanish: $H^\infty(\partial D)$, $C^+(\partial D)$, $A^+(\partial D)$. It is also possible to interpret these subalgebras as algebras of analytic functions inside of the unit disk. We refer to [8] for these and all further details concerning these algebras.

If T is any contraction on a Banach space X, it is possible to define an operator f(T) for any $f \in A^+(\partial D)$ by $f(T) := \sum_{n \in \mathbb{N}_0} \hat{f}(n) T^n$. For certain contractions on a Hilbert space a $H^\infty(\partial D)$–functional–calculus has been defined by B. Sz.-Nagy and C. Foias [15]. An important ingredient of such extensions is given by von Neumann's inequality $\|p(T)\| \leq \|p\|_\infty$, valid for all contractions on a Hilbert space and all polynomials $p \in A^+(\partial D)$. More generally a contraction T on a Banach space is called polynomially bounded, if $\|p(T)\| \leq K\|p\|_\infty$ is valid for a constant $K \geq 0$ and all polynomials $p \in A^+(\partial D)$.

The relations between functional calculi and polynomial boundedness can be stated in a concise way if the following local version of the concept of polynomial boundedness is introduced, which to the knowledge of the author has not been systematically investigated up to now.

Definition 2.1 *Let T be a contraction on a Banach space X. T is called (locally) polynomially bounded in a vector $x \in X$, if*

$$\|p(T)x\| \leq K_x \|p\|_\infty$$

is valid for a constant $K_x \geq 0$ and all polynomials $p \in A^+(\partial D)$.

If they exist we shall assume K or K_x to be chosen minimally. We shall also speak of a polynomially bounded vector $x \in X$, with respect to the operator T, in the sense defined above.

Remark 2.1: A contraction T is (globally) polynomially bounded if and only if it is (locally) polynomially bounded in all $x \in X$.

Proof: Apply the principle of uniform boundedness to the set $\{p(T) : \|p\|_\infty = 1\}$.

Definition 2.2 *Let T be a contraction on a Banach space X and let $\mathcal{A}$ be a topological algebra of functions on ∂D which contains all polynomials $p \in A^+(\partial D)$.*

A local $\mathcal{A}$–functional–calculus of T in $x \in X$ is a continuous linear map $\mathcal{F} : \mathcal{A} \to X$ with the following properties:
(L1) $\mathcal{F}(p) = p(T)x$ for all polynomials $p \in A^+(\partial D)$.
(L2) $\mathcal{F}(pf) = p(T)\mathcal{F}(f)$ for all polynomials $p \in A^+(\partial D)$ and all $f \in \mathcal{A}$.

We shall write f(T)x instead of $\mathcal{F}(f)$, which is consistent with the usual interpretation of this symbol for polynomials by (L1). Now (L2) reads: $(pf)(T)x = p(T)(f(T)x)$ for all $p \in A^+(\partial D)$ and $f \in \mathcal{A}$.

If we have a local $\mathcal{A}$–functional–calculus $\mathcal{F}$ of T in $x \in X$, then for all $g \in \mathcal{A}$ the map

$$\mathcal{F}_g : \mathcal{A} \ni f \mapsto (fg)(T)(x) \in X$$

defines a local $\mathcal{A}$–functional–calculus of T in g(T)x: Continuity and linearity follow from (L1) for $\mathcal{F}$ and from continuity and linearity of the multiplication in $\mathcal{A}$. If $p \in A^+(\partial D)$ is a polynomial and $f \in \mathcal{A}$, then
$\mathcal{F}_g(p) = (pg)(T)x = p(T)(g(T)x)$,
$\mathcal{F}_g(pf) = (pfg)(T)x = p(T)((fg)(T)x) = p(T)(\mathcal{F}_g(f))$,
which proves (L1) and (L2) for $\mathcal{F}_g$.

Now in the obvious way the equation $f(T)(g(T)x) = (fg)(T)x$ is well defined and true, which justifies the terminus "local functional calculus". Indeed if we have consistent $\mathcal{A}$–functional–calculi simultaneous for all $x \in X$, then this is just a functional calculus in the usual sense: a continuous homomorphism of algebras.

In our first theorem we consider $C^+(\partial D)$–functional–calculi.

Theorem 2.1 *Let T be a contraction on a Banach space X and $x \in X$. The following assertions are equivalent:*
(1) T is (locally)polynomially bounded in x.
(2) There is a (norm continuous) local $C^+(\partial D)$–functional–calculus of T in x.
(3) For every $x^* \in X^*$ (the dual of X) there is a measure $\mu_{x,x^*} \in \mathcal{M}[0, 2\pi)$ with

$$< T^n x, x^* >= \int_0^{2\pi} e^{int}\, d\mu_{x,x^*}$$

for all $n \in \mathbb{N}_0$.

Proof:

$$(1) \Rightarrow (2)$$

For any $f \in C^+(\partial D)$ there is a sequence $\{p_n\}_{n\in\mathbb{N}} \subset C^+(\partial D)$ of polynomials such that $\|p_n - f\| \to 0$. By (1) the sequence $\{p_n(T)x\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X. We define f(T)x to be the limit of this sequence. It is now easy to see that this limit does not depend on the choice of the polynomials and that the map $f \mapsto f(T)x$ is normcontinuous. (L1) is valid by definition and (L2) follows from
$p(T)(f(T)x) = p(T)(\lim_{n\to\infty} p_n(T)x) = \lim_{n\to\infty} p(T)p_n(T)x$
$= \lim_{n\to\infty}(pp_n)(T)x = (pf)(T)x.$

$$(2) \Rightarrow (3)$$

The map $C^+(\partial D) \ni f \mapsto < f(T)x, x^* >$ is a continuous linear functional which can be extended to $C(\partial D)$ by the Hahn-Banach theorem. This functional corresponds to a Borel measure whose image μ_{x,x^*} by the parametrization of ∂D satisfies the condition above.

$$(3) \Rightarrow (1)$$

Regard $\{p(T)x : \|p\|_\infty = 1\}$ as a set of linear functionals on X^*. We have $< p(T)x, x^* >= \int_0^{2\pi} p(e^{it})\, d\mu_{x,x^*} \leq \|p\|_\infty \|\mu_{x,x^*}\|$. Now the principle of uniform boundedness implies $K_x = \sup_{\|p\|_\infty = 1} \|p(T)x\| < \infty$.

Remark 2.2: We can also conclude that f(T)x is also polynomially bounded and $K_{f(T)x} \leq K_x \|f\|_\infty$.

Remark 2.3: The measure μ_{x,x^*} is obviously not unique. By the theorem of F. and M.Riesz [8, p.47] the difference between two choices is absolutely continuous with respect to Lebesgue measure.

We consider now local $H^\infty(\partial D)$–calculi. Remember that $H^\infty(\partial D)$ carries a weak* topology inherited from the duality (L^1, L^∞).

Theorem 2.2 *Let X be a Banach space which carries a weak* topology induced by a predual X_* and let T be a weak* continuous contraction on X and $x \in X$.*
The following assertions are equivalent:
(1) If a sequence $\{p_n\}_{n \in \mathbb{N}} \subset H^\infty(\partial D)$ of polynomials is weak Cauchy, then the sequence $\{p_n(T)x\}_{n \in \mathbb{N}}$ is also weak* Cauchy.*
(2) There is a (weak continuous) local $H^\infty(\partial D)$–functional–calculus of T in x.*
(3) For every $\psi \in X_*$ there is a function $g_{x,\psi} \in L^1[0, 2\pi)$ with

$$< T^n x, \psi >= \int_0^{2\pi} e^{int} g_{x,\psi}(t)\, dt \text{ for all } n \in \mathbb{N}_0.$$

(4) $\lim_{n\to\infty}[\sup_{\substack{p \ Pol. \\ \|p\|_\infty \leq 1}} | < p(T)T^n x, \psi > |] = 0$ for all $\psi \in X_*$.

Proof:

$$(1) \Rightarrow (2)$$

For any $f \in H^\infty(\partial D)$ choose a sequence $\{p_n\}_{n \in \mathbb{N}} \subset H^\infty(\partial D)$ of polynomials which converges to f in the weak* topology. By (1) the sequences $\{< p_n(T)x, \psi >\}_{n \in \mathbb{N}}$ are convergent for all $\psi \in X_*$.
By the principle of uniform boundedness the sequence $\{p_n(T)x\}_{n \in \mathbb{N}}$ converges in the weak* topology to an element of $(X_*)^* = X$ which we call f(T)x. Approximation by polynomials shows that this definition of f(T)x

is independent of the choice of sequences and that the map $f \mapsto f(T)x$ is sequentially weak* continuous. The complete weak* continuity follows from that because of the separability of the predual of $H^\infty(\partial D)$ which can be identified with a quotient of $L^1(\partial D)$. This is a general fact for weak* topologies [2]. (L1) is true by definition. Because T and therefore also p(T) are weak* continuous, we can prove (L2) by
$p(T)(f(T)x) = p(T)(\lim_{n\to\infty} p_n(T)x) = \lim_{n\to\infty} p(T)p_n(T)x$
$= \lim_{n\to\infty}(pp_n)(T)x = (pf)(T)x,$
where now the limits are weak*–limits.

$$(2) \Rightarrow (3)$$

The weak* continuity of the map $\mathcal{F} : f \mapsto f(T)x$ implies the existence of a preadjoint $\mathcal{F}_* : X_* \mapsto H^\infty(\partial D)_*$. Again $H^\infty(\partial D)_*$ may be identified with a quotient of $L^1(\partial D)$. For any $\psi \in X_*$ we choose $g_{x,\psi}$ to be a representative of the equivalence class of $\mathcal{F}_*\psi$ (more precisely we still have to transfer it to $[0, 2\pi)$ via parametrization).
For any $f \in H^\infty(\partial D)$ we have

$$< f(T)x, \psi >=< \mathcal{F}(f), \psi >=< f, \mathcal{F}_*\psi >=< f, g_{x,\psi} >= \int_0^{2\pi} f(e^{it})g_{x,\psi}(t)\, dt.$$

Choose $f(e^{it}) = e^{int}$ to get (3).

$$(3) \Rightarrow (1)$$

Let $\{p_n\}_{n\in\mathbb{N}} \subset H^\infty(\partial D)$ be a weak* Cauchy sequence of polynomials. For all $\psi \in X_*$ we have $< p_n(T)x, \psi >= \int_0^{2\pi} p_n(e^{it})g_{x,\psi}(t)\, dt$. The conclusion follows by the defining property of the weak* topology of $H^\infty(\partial D)$.

$$(3) \Rightarrow (4)$$

By (3) there is a function $g_{x,\psi}(t) \in L^1[0, 2\pi)$ with
$< p(T)T^n x, \psi >= \int_0^{2\pi} p(e^{it})e^{int}g_{x,\psi}(t)\, dt$ for all polynomials $p \in H^\infty(\partial D)$.
For every $\epsilon > 0$ there is a number $M \in \mathbb{N}$ and a function $\tilde{g} \in L^1[0, 2\pi)$ with $\tilde{g}(t) = \sum_{k=-M}^{M} a_k e^{ikt}$ and $\|g_{x,\psi} - \tilde{g}\| \leq \epsilon$. For example take Césaro means of partial sums of the Fourier expansion of $g_{x,\psi}$.
For all $n \geq M + 1$ we have

$$\left| \int_0^{2\pi} p(e^{it})e^{int}g_{x,\psi}(t)\, dt \right| = \left| \int_0^{2\pi} p(e^{it})e^{int} \left[\tilde{g}(t) + (g_{x,\psi} - \tilde{g})(t)\right] dt \right|$$

$$\leq\ 0 + \|p\|_\infty \|g_{x,\psi} - \tilde{g}\|_1 \leq \|p\|_\infty \epsilon$$

and therefore $\sup_{\substack{p \ Pol. \\ \|p\|_\infty \leq 1}} | < p(T)T^n x, \psi > | \leq \epsilon$. This proves (4).

(4) $\Rightarrow$ (1)

Let $\{p_n\}_{n\in\mathbb{N}} \subset H^\infty(\partial D)$ be a weak* Cauchy sequence of polynomials. For every $M \in \mathbb{N}$, $\epsilon > 0$ the set

$$U_\epsilon^M := \{g \in H^\infty(\partial D) : |\hat{g}(k)| \leq \frac{\epsilon}{M} \text{ for } k = 0, \ldots, M-1\}$$

is a weak* neighbourhood of 0 in $H^\infty(\partial D)$ because the maps
$g \mapsto \hat{g}(k)$ $(k \in \mathbb{N}_0)$ are weak* continuous linear functionals.
If $g = p_n - p_m$ we have for all $\psi \in X_*$ and all $M \in \mathbb{N}$:

$$| < g(T)x, \psi > | = | < \sum_{k=0}^{M-1} \hat{g}(k) T^k x, \psi > + < \sum_{k=M}^{\infty} \hat{g}(k) T^k x, \psi > |$$

$$\leq \sum_{k=0}^{M-1} |\hat{g}(k)| \, \|x\| \, \|\psi\| + \ \| \sum_{k=M}^{\infty} \hat{g}(k) e^{ikt} \|_\infty \sup_{\substack{p \ Pol. \\ \|p\|_\infty \leq 1}} | < p(T) T^M x, \psi > |$$

$$\leq \sum_{k=0}^{M-1} |\hat{g}(k)| \, \|x\| \, \|\psi\| + \ (\|g\|_\infty + \sum_{k=0}^{M-1} |\hat{g}(k)|) \sup_{\substack{p \ Pol. \\ \|p\|_\infty \leq 1}} | < p(T) T^M x, \psi > |.$$

By (4) it is possible to obtain $\sup_{\substack{p \ Pol. \\ \|p\|_\infty \leq 1}} | < p(T)T^M x, \psi > | \leq \epsilon$ by choosing $M \in \mathbb{N}$ large enough. Take now also $n_0 \in \mathbb{N}$ large enough so that for all $n, m \geq n_0$ we have $p_n - p_m \in U_\epsilon^M$, in particular $\sum_{k=0}^{M-1} |\hat{g}(k)| \leq \epsilon$.
Summarizing we have for all $n, m \geq n_0$:

$$| < (p_n - p_m)(T)x, \psi > | \leq\ \epsilon \, \|x\| \, \|\psi\| + (2 \sup_{k\in\mathbb{N}} \|p_k\|_\infty + \epsilon) \, \epsilon$$

($\sup_{k\in\mathbb{N}} \|p_k\|_\infty < \infty$ because $\{p_n\}_{n\in\mathbb{N}}$ is weak* Cauchy). This implies that $\{p_n(T)x\}_{n\in\mathbb{N}}$ is weak* Cauchy.

Remark 2.4: A global version of (4) (i.e. simultaneous for all $x \in X$) was discussed in a somewhat different context in [1]. (4) should be interpreted as a version of the Riemann–Lebesgue lemma on the Fourier coefficients of L^1–functions. See the following

Proposition: Suppose $\mu \in \mathcal{M}[0, 2\pi)$.
The following assertions are equivalent:
(i) μ is absolutely continuous (with respect to Lebesgue measure).
(ii) $\lim_{n\to\infty}[\sup_{\substack{p\ Pol.\\ \|p\|_\infty \le 1}} \int_0^{2\pi} p(e^{it})e^{int}\, d\mu\,] = 0$.

Proof: Regard the multiplication operator $M_{(\exp it)}$ on $L^2([0, 2\pi), \mu)$ and choose $x = \psi = 1$ to apply Theorem 2.2. We only have to observe that the steps (3) $\Rightarrow$ (4) and (4) $\Rightarrow$ (1) in the proof of the theorem are valid for every single ψ.
Considering this the implication $(i) \Rightarrow (ii)$ follows immediately from (3) $\Rightarrow$ (4).
To prove the implication $(ii) \Rightarrow (i)$, let $\{p_n\}_{n\in\mathbb{N}}$ be a weak* Cauchy sequence of polynomials. (4) $\Rightarrow$ (1) shows that the sequence
$\{< p_n(T)1, 1 >\}_{n\in\mathbb{N}} = \{\int_0^{2\pi} p_n(e^{it})\, d\mu\}_{n\in\mathbb{N}}$ is convergent. We infer that the map $p \mapsto \int_0^{2\pi} p(e^{it})\, d\mu$ extends to a weak* continuous linear functional on $H^\infty(\partial D)$. If we represent this functional by an absolutely continuous measure, we know (see Remark 2.3) that its difference from μ also has to be absolutely continuous.

Definition 2.3 *If the conditions of Theorem 2.2 are satisfied the operator T is called absolutely continuous in $x \in X$. We shall also call $x \in X$ an absolutely continuous vector with respect to T in this case.*
$X_{pb} := \{x \in X :$ *T is polynomially bounded in x* $\}$,
$X_{ac} := \{x \in X :$ *T is absolutely continuous in x* $\}$.

We call T (globally) absolutely continuous if $X = X_{ac}$. This is consistent with the definition in [13]. Whenever we speak of absolute continuity we shall implicitly assume that the setting of Theorem 2.2 is given.
X_{pb} and X_{ac} are linear subspaces of X, and we have $X_{ac} \subset X_{pb}$ because the weak* continuity of the local functional calculus $\mathcal{F}$ implies its norm continuity. Therefore the property of local polynomial boundedness is a necessary condition for more advanced calculi. This has been our first motivation for a detailed study of this concept.

Remark 2.5: If $x \in X$ is absolutely continuous with respect to T then $T^n x \to 0$ in the weak* sense. This follows from Theorem 2.2(3) and the Riemann-Lebesgue lemma. A partial converse is the following: If T

is (globally) polynomially bounded and $\|T^n x\| \to 0$, then x is absolutely continuous.
This follows immediately from Theorem 2.2(4).

Many well known facts about Hilbert space contractions fit naturally into this frame:
It is easy to see that a vector is absolutely continuous in the sense defined above if and only if its spectral measure with respect to a unitary dilation is absolutely continuous. Because for completely nonunitary contractions all these spectral measures are absolutely continuous [15], we can obtain the $H^\infty(\partial D)$–calculus of B. Sz.-Nagy and C. Foias mentioned in the beginning from our Theorem 2.2. We give another result which also generalizes some well known facts of Hilbert space theory:

Proposition: Let T be a contraction on a Banach space X.
(a) The map $x \mapsto K_x$ is a norm on X_{pb}.
(b) If X_{pb} is (norm-)closed so is X_{ac}.

Proof: Remember that we agreed to choose K_x minimally. Thus K_x is the norm of the operator $C^+(\partial D) \ni p \mapsto p(T)x \in X$. This proves (a).
We now prove (b): If X_{pb} is closed then $T|_{X_{pb}}$ is (globally) polynomially bounded: there is a number $K \geq 0$ with $\|p(T|_{X_{pb}})\| \leq K\|p\|_\infty$ (see Remark 2.1). Again we choose K minimally. For all $x \in X_{pb}$ we have
$\|p(T)x\| \leq K\,\|x\|\,\|p\|_\infty$ and therefore $K_x \leq K\|x\|$ (so the original norm is finer than the one defined in part (a)).
Let now $x \in X$ be in the (norm-)closure of X_{ac}. By assumption we have $x \in X_{pb}$. Take a weak* Cauchy sequence $\{p_n\}_{n\in\mathbb{N}}$ of polynomials and $\epsilon > 0$. Fix now $\psi \in X_*$ and choose $y \in X_{ac}$ with $\|x-y\|$ small enough so that:

$$| < (p_n - p_m)(T)(x-y), \psi > | \leq 2 \sup_{n\in\mathbb{N}} \|p_n\|_\infty\, K\,\|x-y\|\,\|\psi\| \leq \frac{1}{2}\epsilon.$$

On the other hand by Theorem 2.2(1) there is a $n_0 \in \mathbb{N}$ so that for all $n, m \geq n_0$ we have $| < (p_n - p_m)(T)y, \psi > | \leq \frac{1}{2}\epsilon$ and thus
$| < (p_n - p_m)(T)x, \psi > | \leq \epsilon$. This proves $x \in X_{ac}$.

In the case of Hilbert space contractions (where $X = X_{pb}$ by von Neumann's inequality) it is a well known result that X_{ac} is closed.

Examples:

Let T be a contraction on a Banach space X.

(a) For an eigenvector x of T^n with eigenvalue λ, $|\lambda| = 1$, we have $x \in X_{pb} \setminus X_{ac}$ and $K_x \leq n\|x\|$.

Proof: Let $p(z) = \sum_{j=0}^{J} \gamma_j z^j$ be a polynomial. Then
$p(T)x = (\gamma_0 + \gamma_n\lambda + \gamma_{2n}\lambda^2 + \ldots)x + (\gamma_1 + \gamma_{n+1}\lambda + \gamma_{2n+1}\lambda^2 + \ldots)Tx + \ldots + (\gamma_{n-1} + \gamma_{2n-1}\lambda + \ldots)T^{n-1}x$.
We have $|\gamma_0 + \gamma_n + \gamma_{2n} + \ldots| = |\frac{1}{n}\sum_{k=0}^{n-1} p(e^{2\pi i\frac{k}{n}})| \leq \|p\|_\infty$,
and by evaluating similar sums we also get
$|\gamma_j + \gamma_{n+j}\lambda + \gamma_{2n+j}\lambda^2 + \ldots| \leq \|p\|_\infty$ for $0 \leq j \leq n-1$.
Now it is easy to establish the assertions in (a).

(b) If the spectral radius of T is strictly smaller than 1, then T is absolutely continuous.

Proof: We can define a $H^\infty(\partial D)$–functional–calculus as a subcalculus of the usual Dunford–calculus.

(c) Let (Ω, Σ, μ) be a measure space and $T = M_f$ a multiplication operator with a function $f \in L^\infty(\Omega, \Sigma, \mu)$ on $X = L^q(\Omega, \Sigma, \mu)$, $1 \leq q \leq \infty$.
If $\|f\|_\infty \leq 1$ then M_f is polynomially bounded.
If $1 < q \leq \infty$ and $|f| < 1$ μ–a.e., then M_f is absolutely continuous.

Proof: $\|p(M_f)x\|_q = \|(p \circ f) \cdot x\|_q \leq \|p\|_\infty \|x\|_q$ proves the first part.
For the second part we define $\mathcal{F} : H^\infty \mapsto L^q$ by $\mathcal{F}(h) := h(M_f)x := (h \circ f) \cdot x$ for all $h \in H^\infty$ (because $|f| < 1$ μ–a.e. this is defined μ–a.e.).
We now show that $\mathcal{F}$ is indeed a H^∞–functional–calculus. For the nontrivial part it suffices to prove: If $\{h_n\}_{n\in\mathbb{N}} \subset H^\infty(\partial D)$ converges to 0 in the weak* sense then $\{h_n(M_f)x\}_{n\in\mathbb{N}} \subset L^q$ also converges to 0 in the weak* sense.
Suppose $\frac{1}{q} + \frac{1}{r} = 1$ and $\psi \in L^r$.
Then $< h_n(M_f)x, \psi >= \int_\Omega h_n(f(\omega))x(\omega)\psi(\omega)\, d\mu(\omega)$. If $|f(\omega)| < 1$ then $\lim_{n\to\infty} h_n(f(\omega)) = 0$, and this is the case μ–a.e.. Because the integrand is dominated by $\sup_{n\in\mathbb{N}} \|h_n\|_\infty |x(\omega)\psi(\omega)| \in L^1$, the assertion follows by Lebesgue's theorem of dominated convergence.
Note that for q=2 this is exactly the case of complete nonunitarity.

3 Relations to Ergodic Theory

In this section we shall examine certain contractions on L^∞–spaces and C^*–algebras and develop methods to prove the existence of vectors which are not polynomially bounded. We shall see that this problem is closely related to the ergodic theory of these operators.

Let (Ω, Σ, μ) be a probability space and T an endomorphism of the algebra $L^\infty(\Omega, \Sigma, \mu)$ which is induced by a measure preserving transformation $\tau : \Omega \mapsto \Omega$, i.e. $Tf(\omega) := f(\tau\omega)$ for all $f \in L^\infty(\Omega, \Sigma, \mu)$.
(If (Ω, Σ, μ) is a Lebesgue space, every automorphism is induced as above, see [11, Theorem 1.4.7].)
If $\tau^n \omega = \omega$ and $n \in \mathbb{N}$ is minimal with that property, then we say that τ has period n in ω. Define $\Omega_n := \{\omega \in \Omega : \tau^n\omega = \omega\}$, $n \in \mathbb{N}$.
The (global) period of τ is equal to n if $\mu(\Omega \setminus \Omega_n) = 0$ and n is minimal with that property. τ is called aperiodic if $\mu(\bigcup_{n\in\mathbb{N}} \Omega_n) = 0$.
There is an obvious decomposition of an arbitrary measure preserving transformation into periodic parts with certain periods and an aperiodic part which can be examined separately: A periodic part of an endomorphism is polynomially bounded. This can be subsumed under example (a) in the last section. For the aperiodic case we need the

Rohlin Lemma: (see [6])
If τ is aperiodic then there is for every $J \in \mathbb{N}$ and $\epsilon > 0$ a set $E \in \Sigma$ such that $E, \tau E, \tau^2 E, \ldots, \tau^J E$ are pairwise disjoint and
$\mu(E \cup \tau E \cup \tau^2 E \cup \ldots \cup \tau^J E) > 1 - \epsilon$.

This result of ergodic theory translates directly into a result about polynomial boundedness:

Theorem 3.1 *Let τ be an aperiodic measure preserving transformation of the probability space (Ω, Σ, μ) and T the induced endomorphism of $L^\infty(\Omega, \Sigma, \mu)$. For any polynomial $p(z) = \sum_{j=0}^J \gamma_j z^j$ we have $\|p(T)\| = \sum_{j=0}^J |\gamma_j| = \|p\|_A$. In particular: T is not (globally) polynomially bounded.*

Proof: If J is the degree of a polynomial p given as above, then by the Rohlin lemma there is a set $E \in \Sigma$ with $\mu(E) > 0$ and $E, \tau E, \tau^2 E, \ldots, \tau^J E$ pairwise disjoint. We define a function
$f \in L^\infty(\Omega, \Sigma, \mu)$ (with $\|f\|_\infty = 1$) by

$$f(\omega) := \begin{cases} \frac{\overline{\gamma_j}}{|\gamma_j|} & \text{if } \omega \in \tau^j E \ (j = 0, \ldots, J) \\ 0 & \text{elsewhere} \end{cases}$$

Then we have for every $\omega \in E$:

$$p(T)f(\omega) = \sum_{j=0}^{J} \gamma_j T^j f(\omega) = \sum_{j=0}^{J} \gamma_j f(\tau^j \omega) = \sum_{j=0}^{J} |\gamma_j|.$$

We conclude that $\|p(T)\| \geq \sum_{j=0}^{J} |\gamma_j|$. The reversed inequality is just the triangle inequality of the norm.

Important examples of aperiodic transformations are given by ergodic transformations on probability spaces without atoms (see [11]), for example rotations of the unit circle with irrational angles. It is interesting to reinterpret Theorem 3.1 in this case from the point of view of Fourier analysis:

Let T be the automorphism of $L^\infty(\partial D)$ induced by $\tau z := e^{2\pi i\alpha} z$ with α irrational. For any $f \in A(\partial D)$ an eigenvector expansion shows $K_f \leq \|f\|_A$ (the norm of the Wiener algebra). In particular: Any $f \in A(\partial D)$ is polynomially bounded with respect to T. On the other hand we can prove Theorem 3.1 directly in this special case without using the Rohlin lemma, by defining the set E to be a small segment of the circle. This shows that the function f defined in the proof of Theorem 3.1 can be extended continuously (instead of the extension by 0 used there). We conclude that even $T|_{C(\partial D)}$ is not polynomially bounded and (using Remark 2.1) there is a function $f \in C(\partial D)$ which is not polynomially bounded with respect to T. We give an application:

Proposition: There is a function $f \in C(\partial D) \setminus A(\partial D)$ with the following property: For any $M \in \mathbb{N}$ there is an almost periodic sequence $\{\rho_n\}_{n \in \mathbf{Z}} \subset \mathbb{C}$ with $|\rho_n| \leq 1$ for all $n \in \mathbf{Z}$, so that the function g with Fourier coefficients $\hat{g}(n) = \hat{f}(n)\rho_n$ for $n \in \mathbf{Z}$ belongs to $C(\partial D) \setminus A(\partial D)$ and $\|g\|_\infty > M$.

Proof: Choose a function $f \in C(\partial D) \setminus A(\partial D)$ which is not polynomially bounded with respect to the irrational rotation (see the discussion preceding

the proposition). If p is any polynomial we get $(p(T)f)\hat{}(n) = \hat{f}(n)p(e^{2\pi i\alpha n})$ for all $n \in \mathbf{Z}$. If any $M \in \mathbb{N}$ is fixed, we can find a polynomial p with $\|p\|_\infty \leq 1$ and $\|p(T)f\|_\infty > M$. Now define $\rho_n := p(e^{2\pi i\alpha n})$ for all $n \in \mathbf{Z}$ and g := p(T)f.
Open question: Is this valid for all $f \in C(\partial D) \setminus A(\partial D)$?

A version of the Rohlin lemma was proved by A.Connes in the context of W^*–algebras. For the terminology of this mathematical topic which is not explicitly defined here we refer to [14].

Let $\mathcal{A}$ be a finite W^*–algebra and μ a faithful normal trace on $\mathcal{A}$ with $\mu(1) = 1$. For any $x \in \mathcal{A}$ we have the C^*–norm $\|x\|$ and the norm $\|x\|_2 := \mu(x^*x)^{\frac{1}{2}}$. A family of pairwise orthogonal and nonvanishing projections $\{p_i\} \subset \mathcal{A}$ with $\sum_i p_i = 1$ is called a partition of unity.

A *–automorphism $T : \mathcal{A} \mapsto \mathcal{A}$ is called aperiodic if there is no projection $p \in \mathcal{A}$ with $T^n|_{p\mathcal{A}p}$ inner for any $n \in \mathbb{N}$.
(If $\mathcal{A} = L^\infty(\Omega, \Sigma, \mu)$ this coincides with the definition above because in this case an inner automorphism acts as identity. We also have the following generalization: If $\mathcal{A}$ is a W^*–algebra without minimal projections and $T : \mathcal{A} \mapsto \mathcal{A}$ is an ergodic *–automorphism then T is aperiodic, see [14, Prop.17.7].) We can now cite the result of A.Connes:

Lemma (Rohlin-Connes) (see [14, 17.17])
Let $\mathcal{A}$ be a finite W^*–algebra , μ a faithful normal trace on $\mathcal{A}$ with $\mu(1) = 1$ and $T : \mathcal{A} \mapsto \mathcal{A}$ an aperiodic *–automorphism with $\mu \circ T = \mu$.
For every $J \in \mathbb{N}$ and $\epsilon > 0$ there is a partition of unity $\{p_0, p_1, \dots, p_J\}$ in $\mathcal{A}$ such that $\|T(p_0) - p_1\|_2 \leq \epsilon$, $\|T(p_1) - p_2\|_2 \leq \epsilon, \dots,$
$\|T(p_{J-1}) - p_J\|_2 \leq \epsilon$, $\|T(p_J) - p_0\|_2 \leq \epsilon$.

With the help of this lemma we can now prove the following analogue of Theorem 3.1:

Theorem 3.2 *Let $\mathcal{A}$ be a finite W^*–algebra , μ a faithful normal trace on $\mathcal{A}$ with $\mu(1) = 1$ and $T : \mathcal{A} \mapsto \mathcal{A}$ an aperiodic *–automorphism with $\mu \circ T = \mu$. For any polynomial $p(z) = \sum_{j=0}^J \gamma_j z^j$ we have $\|p(T)\| = \sum_{j=0}^J |\gamma_j| = \|p\|_A$. In particular: T is not (globally) polynomially bounded.*

Proof: Let the polynomial $p(z) = \sum_{j=0}^{J} \gamma_j z^j$ be given. For any fixed $\delta > 0$ choose $\epsilon > 0$ such that $(\sum_{j=0}^{J} |\gamma_j| j)(J+1)^{\frac{3}{2}}\epsilon \leq \delta$, and for J, ϵ let a partition of unity according to the lemma of Rohlin–Connes be fixed. We may assume $\|p_0\|_2 \geq (J+1)^{-\frac{1}{2}}$ because we have $\sum_{j=0}^{J} \|p_j\|_2^2 = 1$.
Define now $f := \sum_{j=0}^{J} \alpha_j p_j$ where the complex coefficients $\{\alpha_j\}_{j=0}^{J}$ are given by the equations

$\gamma_0 \alpha_0 = |\gamma_0|$,

$\gamma_j \alpha_{J+1-j} = |\gamma_j|$, $j = 1, \ldots, J$.

We have $f \in \mathcal{A}$ and $\|f\| = 1$. We can now perform the following computation where we indicate in brackets the justification of the main steps:

$$\|\sum_{j=0}^{J} \gamma_j (T^j f) p_0 - \sum_{j=0}^{J} |\gamma_j|\, p_0\|_2 \leq \sum_{j=0}^{J} \|\gamma_j \sum_{k=0}^{J} \alpha_k (T^j p_k) p_0 - |\gamma_j|\, p_0\|_2$$

$$\leq \sum_{j,k=0}^{J} \|\gamma_j \alpha_k (T^j p_k) p_0 - \gamma_j \alpha_k \, p_{(j+k)mod(J+1)}\, p_0\|_2$$

[Because of the orthogonality of the projections in the partition of unity we have $p_{(j+k)mod(J+1)}\, p_0 = 0$ if (j+k)mod(J+1) $\neq$ 0.]

$$\leq \sum_{j,k=0}^{J} |\gamma_j| j\epsilon$$

[Here we used ihe inequality $\|T^j p_k - p_{(j+k)mod(J+1)}\|_2 \leq j\epsilon$ for $j, k = 0, \ldots, J$ which follows from the lemma of Rohlin-Connes by induction.]

$$= (\sum_{j=0}^{J} |\gamma_j| j)(J+1)\epsilon \;\leq\; \delta(J+1)^{-\frac{1}{2}}.$$

We can conclude from that

$$\|p(T)f\|\, \|p_0\|_2 \geq \|p(T)f \cdot p_0\|_2 \geq \sum_{j=0}^{J} |\gamma_j|\, \|p_0\|_2 - \delta(J+1)^{-\frac{1}{2}} \geq (\sum_{j=0}^{J} |\gamma_j| - \delta)\, \|p_0\|_2.$$

Considering arbitrarily small values of δ we get the inequality $\|p(T)\| \geq \sum_{j=0}^{J} |\gamma_j|$. The reversed inequality is the triangle inequality of the norm.

Remark 3.1: In particular we can infer immediately that the Banach algebra in $\mathcal{B}(\mathcal{A})$, the bounded operators on $\mathcal{A}$, which is generated by $\{T^n\}_{n \in \mathbf{Z}}$ is isomorphic to $l^1(\mathbf{Z})$. The (Gelfand-)spectrum of this algebra is ∂D, and we conclude that the spectrum of T in $\mathcal{B}(\mathcal{A})$ is ∂D (see [14, 14.12]).

The algebra $l^1(\mathbf{Z})$ (isomorphic to the Wiener algebra $A(\partial D)$ via Fourier transform) which made their appearance in the last remark as well as its subalgebra $l^1(\mathbb{N}_0)$ (isomorphic to $A^+(\partial D)$) will play a prominent role from now on. We shall need some well known results about embeddings of $l^1(\mathbb{N}_0)$ into Banach spaces which we summarize now for the convenience of the reader. Proofs for the facts quoted below may be found in [3], in particular in chapter XI.

A basic sequence in a Banach space is a sequence which is a Schauder base for its closed linear span.

A bounded basic sequence $\{x_j\}_{j\in\mathbb{N}_0}$ is called equivalent to the unit vector base $\{e_j\}_{j\in\mathbb{N}_0}$ ($e_j(k) = \delta_{jk}$) of $l^1(\mathbb{N}_0)$ if there is a constant $c > 0$ so that

$$c\sum_{j=0}^{J}|\gamma_j| \leq \|\sum_{j=0}^{J}\gamma_j x_j\|$$

for all $(\gamma_0, \gamma_1, \ldots, \gamma_J) \in \mathbb{C}^{J+1}$ and all $J \in \mathbb{N}_0$.
This is just a necessary and sufficient condition for the map $x_j \mapsto e_j$ ($j \in \mathbb{N}_0$) to extend linearly to a topological isomorphism of the corresponding closed linear spans.

The following method to identify basic sequences in $l^\infty(\Omega)$ (where Ω is any set) which are equivalent to the unit vector base of $l^1(\mathbb{N}_0)$ is due to H.P.Rosenthal and L.Dor in connection with their embedding theorem. First we have to prepare some terminology:
A sequence of subsets $\{\Omega_n\}_{n\in\mathbb{N}}$ of Ω is called a tree if $\Omega_1 = \Omega$ and if for all $n \in \mathbb{N}$ the sets Ω_{2n} and Ω_{2n+1} are disjoint subsets of Ω_n.
A sequence $\{(E_j, O_j)\}_{j\in\mathbb{N}_0}$ of pairs of disjoint subsets of Ω is called independent if for any given disjoint finite subsets $\mathcal{P}, \mathcal{N}$ of $\mathbb{N}_0$ the sets $\bigcap_{j\in\mathcal{P}} E_j$ and $\bigcap_{j\in\mathcal{N}} O_j$ are not disjoint.

Remark 3.2: If a tree $\{\Omega_n\}_{n\in\mathbb{N}}$ of Ω is given then we can form an independent sequence of pairs of disjoint subsets if we define E_j to be the union of all Ω_n with $n = 2^{j+1}, 2^{j+1}+1, \ldots, 2^{j+2}-1$ and n even and O_j to be the union of all Ω_n with $n = 2^{j+1}, 2^{j+1}+1, \ldots, 2^{j+2}-1$ and n odd. This may be interpreted as an imitation of the stochastic independence of Rademacher functions.

Lemma (Rosenthal–Dor) (see [3, chapter XI])
Suppose $\{x_j\}_{j\in\mathbb{N}_0} \subset l^\infty(\Omega)$ to be a bounded sequence, $D_0, D_1 \subset \mathbb{C}$ to be closed disks with diameter d and distance δ, $d \leq \frac{1}{2}\delta$.
If $\{(E_j, O_j)\}_{j\in\mathbb{N}_0}$ is an independent sequence of pairs of disjoint subsets of Ω with $x_j(\omega) \in D_0$ if $\omega \in E_j$ and $x_j(\omega) \in D_1$ if $\omega \in O_j$ for all $j \in \mathbb{N}_0$, then we have

$$\frac{\delta}{8}\sum_{j=0}^{J}|\gamma_j| \leq \|\sum_{j=0}^{J}\gamma_j x_j\|_\infty$$

for all $(\gamma_0, \gamma_1, \ldots, \gamma_J) \in \mathbb{C}^{J+1}$ and all $J \in \mathbb{N}_0$. In particular:
$\{x_j\}_{j\in\mathbb{N}_0}$ is equivalent to the unit vector base of $l^1(\mathbb{N}_0)$.

Remark 3.3: The proof of the lemma given in [3] shows that the following modifications are legitimate:
(a) If we have only finite sets $\{x_j\}_{j=0}^{J}$ and $\{(E_j, O_j)\}_{j=0}^{J}$ with the relations given in the lemma, then for this particular J the inequality remains valid:

$$\frac{\delta}{8}\sum_{j=0}^{J}|\gamma_j| \leq \|\sum_{j=0}^{J}\gamma_j x_j\|_\infty$$

for all $(\gamma_0, \gamma_1, \ldots, \gamma_J) \in \mathbb{C}^{J+1}$.
In this case we adapt the terminology given above in the obvious way.
(b) If we consider the essential supremum norm on $L^\infty(\Omega, \Sigma, \mu)$ instead of $l^\infty(\Omega)$, then we can use an analogous "essential" terminology and prove an "essential" Rosenthal–Dor lemma.

We shall now use the lemma to continue our examination of endomorphisms with respect to polynomial boundedness. Recall that a measure preserving transformation τ on a probability space (Ω, Σ, μ) and its induced endomorphism T of $L^\infty(\Omega, \Sigma, \mu)$ are called weakly mixing if all eigenvectors of T are constants. Weak mixing implies ergodicity, but the converse is not true as can be seen from the irrational rotation considered above (see [11, 2.6]).

Theorem 3.3 *Let (Ω, Σ, μ) be a probability space and T an endomorphism of $X = L^\infty(\Omega, \Sigma, \mu)$ which is induced by a measure preserving transformation τ.*

The following assertions are equivalent:

(1) T is weakly mixing.

(2) Every vector which is polynomially bounded with respect to T is a constant, i.e. $X_{pb} = \mathbb{C}1$.

Proof: All constants are eigenvectors and therefore polynomially bounded (see example (a) in the second section). If T is not weakly mixing then there are other eigenvectors than the constants. It remains to show: If T is weakly mixing and $f \in L^\infty(\Omega, \Sigma, \mu)$ is not constant (a.e.), then f is not polynomially bounded with respect to T.

Because f is not a constant there are at least two different values c_0, c_1 in the essential range of f, i.e. in particular: If $D_0, D_1 \subset \mathbb{C}$ are disks with centers c_0, c_1 and diameter d (where it is possible to choose $d \leq \frac{1}{2}\delta$ as in the lemma of Rosenthal–Dor if δ is the distance of the disks) then we have $\mu(f^{-1}D_i) > 0$, i=0,1. Define $\Sigma \ni A_0 := f^{-1}D_0$, $\Sigma \ni A_1 := f^{-1}D_1$.

Suppose now that p is a polynomial, $p(z) = \sum_{j=0}^{J} \gamma_j z^j$. We shall use a result of H. Furstenberg (see [11, 4.3] or [5, chapter 4]) showing that the following assertion about τ is equivalent to the weak mixing property:
There is a set $\mathcal{J} \subset \mathbb{N}$ of density zero so that for any sets
$C_0, C_1, \ldots, C_k \in \Sigma$:

$$\lim_{m\to\infty, m\notin\mathcal{J}} \mu(C_0 \cap \tau^{-m}C_1 \cap \tau^{-2m}C_2 \cap \ldots \cap \tau^{-km}C_k) = \mu(C_0)\mu(C_1)\ldots\mu(C_k).$$

Using this property of weakly mixing transformations we conclude that there is a number $M = M(J) \in \mathbb{N}$ so that for all possible choices
$C_j \in \{A, B\}$, $j = 0, \ldots, J$ (there are a finite number of possibilities for that; J is the degree of the given polynomial) we always have

$$\mu(C_0 \cap \tau^{-M}C_1 \cap \tau^{-2M}C_2 \cap \ldots \cap \tau^{-jM}C_j) > 0 \ (0 \leq j \leq J).$$

We can now see that the following finite tree of subsets $\{\Omega_n\}_{n=1}^{2^{J+2}-1}$ consists of sets of positive μ–measure (see Remark 3.3):

$\Omega_1 := \Omega$,

$\Omega_2 := A_0$, $\Omega_3 := A_1$,

$\Omega_4 := A_0 \cap \tau^{-M}A_0$, $\Omega_5 := A_0 \cap \tau^{-M}A_1$, $\Omega_6 := A_1 \cap \tau^{-M}A_0$,

$\Omega_7 := A_1 \cap \tau^{-M}A_1$, $\Omega_8 := A_0 \cap \tau^{-M}A_0 \cap \tau^{-2M}A_0, \ldots$ etc.

The general formula for $n = 2, 3, \ldots, 2^{J+2} - 1$:
If $n = 2^{j+1} + 2^j \epsilon_j + 2^{j-1}\epsilon_{j-1} + 2\epsilon_1 + \epsilon_0$ (where $\epsilon_i \in \{0, 1\}$ for all i)
then we have $\Omega_n := A_{\epsilon_j} \cap \tau^{-M} A_{\epsilon_{j-1}} \cap \tau^{-2M} A_{\epsilon_{j-2}} \cap \ldots \cap \tau^{-jM} A_{\epsilon_0}$.

As in Remark 3.2 we can use this tree to define a finite independent sequence $\{(E_j, O_j)\}_{j=0}^J$ of pairs of disjoint sets:

$$E_j := \bigcup_{\substack{n=2^{j+1} \\ n\, even}}^{2^{j+2}-1} \Omega_n = \bigcup_{n=2^{j+1}+\ldots+2\epsilon_1+0} \Omega_n$$

$$O_j := \bigcup_{\substack{n=2^{j+1} \\ n\, odd}}^{2^{j+2}-1} \Omega_n = \bigcup_{n=2^{j+1}+\ldots+2\epsilon_1+1} \Omega_n.$$

If we have $\omega \in E_j$ $(j = 0, \ldots, J)$ then there is a number
$n = 2^{j+1} + 2^j \epsilon_j + 2^{j-1}\epsilon_{j-1} + 2\epsilon_1 + \epsilon_0$ with $\epsilon_0 = 0$ such that $\omega \in \Omega_n$,
in particular $\omega \in \tau^{-jM} A_0$, i.e. $T^{jM} f(\omega) = f(\tau^{jM}\omega) \in D_0$ $(j = 0, \ldots, J)$.
In the same way we conclude $T^{jM} f(\omega) \in D_1$ if $\omega \in O_j$ $(j = 0, \ldots, J)$.

Now we are in a position to apply the Rosenthal–Dor lemma.
Define $\tilde{p}(z) := p(z^M)$. Note that $\|p\|_\infty = \|\tilde{p}\|_\infty$. If we assume f to be polynomially bounded (so we aim to get a contradiction) we get the following inequalities:

$$\frac{\delta}{8} \sum_{j=0}^{J} |\gamma_j| \leq \| \sum_{j=0}^{J} \gamma_j T^{jM} f \|_\infty \leq K_f \, \|\tilde{p}\|_\infty = K_f \, \|p\|_\infty.$$

δ as well as K_f are independent of p. Therefore with $c := \frac{\delta}{8K_f}$ we get for any polynomial p with $p(z) = \sum_{j=0}^{J} \gamma_j z^j$:

$$c \sum_{j=0}^{J} |\gamma_j| \leq \|p\|_\infty \, (\leq \sum_{j=0}^{J} |\gamma_j|).$$

This implies the supremum norm to be equivalent with the norm $\| \cdot \|_A$ of the Wiener algebra on the space of all polynomials. This is clearly false, and we have reached a contradiction.

It is possible to extend the definition of the weak mixing property to arbitrary contractions on $L^\infty(\Omega, \Sigma, \mu)$. A contraction T is weakly mixing if any eigenvector with an eigenvalue whose modulus equals the spectral radius of T is necessarily a constant. The question arises whether Theorem 3.3 may also be generalized. Without further conditions this is of course true for restrictions of endomorphisms. We now prove a less trivial result in this direction with the use of dilation techniques.

Theorem 3.4 *Let (Ω, Σ, μ) be a probability space and $X = L^\infty(\Omega, \Sigma, \mu)$. Suppose T to be a positivity preserving and weakly mixing contraction, $T1 = 1, \mu \circ T = \mu$. Let $f \in L^\infty(\Omega, \Sigma, \mu)$ be decomposed in the form $f = c \cdot 1 + \tilde{f}, c \in \mathbb{C}, \int_\Omega \tilde{f}\, d\mu = 0$.*
If $\lim_{n\to\infty} \|T^n \tilde{f}\|_2 \neq 0$ then the vector f is not polynomially bounded with respect to T. ($\|\cdot\|_2$ is the norm of $L^2(\Omega, \Sigma, \mu)$).

Obviously this is an extension of Theorem 3.3.

Proof: The operator T may be looked upon as a transition operator of a Markov process. From the point of view of operator theory we can speak of a (minimal) Markov dilation, i.e. there is a probability space $(\hat{\Omega}, \hat{\Sigma}, \hat{\mu})$ and on that space a measure preserving transformation $\hat{\tau}$ which induces an automorphism $\hat{T}$ of $\hat{X} := L^\infty(\hat{\Omega}, \hat{\Sigma}, \hat{\mu})$ such that the following diagram commutes for all $n \in \mathbb{N}_0$:

$$\begin{array}{ccccc} & X & \stackrel{T^n}{\rightarrow} & X & \\ j & \downarrow & & \uparrow & P \\ & \hat{X} & \stackrel{\hat{T}^n}{\rightarrow} & \hat{X} & \end{array}$$

Here j denotes an injective algebra homomorphism and $P_0 :=$ jP is a conditional expectation of $\hat{X}$ onto its subalgebra jX, and we have the following additional properties:

– $\int_{\hat{\Omega}} j(f) d\hat{\mu} = \int_\Omega f\, d\mu$ for all $f \in X$.

– If the conditional expectation of $\hat{X}$ onto $\bigvee_{k\in\mathbb{N}_0} \hat{T}^{-k}(jX)$ is denoted by $P_{(-\infty,0]}$, then for all $g \in \bigvee_{k\in\mathbb{N}_0} \hat{T}^k(jX)$ the Markov property $P_{(-\infty,0]}g = P_0 g$ is valid.

– $\bigvee_{k\in\mathbf{Z}} \hat{T}^k(jX) = \hat{X}$ (Minimality)

[Here "$\bigvee$" denotes the W^*–algebraic hull.]

A more detailed discussion of the concept of a (minimal) Markov dilation which we have introduced here in a very short manner can be found in [10]. We shall need the following result in the sequel: If T is weakly mixing then also $\hat{T}$ is weakly mixing (see [12] or [10]).

Returning to the proof of Theorem 3.4 we infer from the preceding discussion that we can apply Theorem 3.3 for $\hat{T}$, and we shall now derive Theorem 3.4 from that. If we form the closures with respect to $\|\cdot\|_2$ in the diagram above and if we use the same symbols for the Hilbert spaces formed in this way, then the diagram describes a (nonminimal) unitary dilation $\hat{T}$ on $L^2(\hat{\Omega}, \hat{\Sigma}, \hat{\mu})$ for the contraction T on $L^2(\Omega, \Sigma, \mu)$.
We infer for all $f \in X$ the existence of the limits $\Lambda f := \lim_{n\to\infty} \hat{T}^{-n} j T^n f$ (with respect to $\|\cdot\|_2$). Indeed, $\Lambda j^{-1}|_{jX}$ is just the orthogonal projection of $\hat{X}$ onto the *–residual part of the dilation restricted to jX. We have used here the terminology of [15, Chap.II.2, II.3], where also a proof for the existence of the limits may be found. (In [15] minimal unitary dilations are examined, but this is not relevant for the result in question.)
From the definition of Λ we get immediately the following properties:
– $\Lambda \in \mathcal{B}(X, \hat{X})$ with $\|\Lambda\| \leq 1$ for $\|\cdot\|_2$ as well as for $\|\cdot\|_\infty$,
– $\Lambda T = \hat{T}\Lambda$, $\Lambda 1 = 1$, $\Lambda(\{1\}^\perp) \subset \{1\}^\perp$,
– $\Lambda f = 0$ is equivalent to $\lim_{n\to\infty} \|T^n f\|_2 = 0$.

To finish the proof we need the following compatibility between polynomial boundedness and similarity of operators which is stated in the

Remark 3.4: Suppose Y,Z to be Banach spaces and $R \in \mathcal{B}(Y)$, $S \in \mathcal{B}(Z)$ to be contractions. If there is an operator $\Lambda \in \mathcal{B}(Y, Z)$ such that $\Lambda R = S\Lambda$, then for any $y \in Y$ which is polynomially bounded with respect to R, the image $\Lambda y \in Z$ is polynomially bounded with respect to S.
If Λ is a similarity between R and S (i.e. $\Lambda R = S\Lambda$ and Λ is invertible) then $y \in Y$ is polynomially bounded with respect to R if and only if $\Lambda y \in Z$ is polynomially bounded with respect to S.

Proof of Remark 3.4: If $y \in Y$ is polynomially bounded with respect to R, i.e. $\|p(R)y\| \leq K_y(R)\|p\|_\infty$ for all polynomials p, then we have $\|p(S)\Lambda y\| = \|\Lambda p(R)y\| \leq \|\Lambda\|\, K_y(R)\, \|p\|_\infty$, therefore $K_{\Lambda y}(S) \leq \|\Lambda\|\, K_y(R)$. The second part follows by applying the first to Λ and Λ^{-1}.

We can now finish the proof of Theorem 3.4: If we have the decomposition $f = c \cdot 1 + \tilde{f}$, $c \in \mathbb{C}$, $\int_\Omega \tilde{f}\, d\mu = 0$ for any $f \in X = L^\infty(\Omega, \Sigma, \mu)$, we conclude (using the properties of Λ listed above) that if

$\lim_{n\to\infty} \|T^n\tilde{f}\|_2 \neq 0$ we get $\Lambda f = c \cdot 1 + \Lambda\tilde{f}$ with $0 \neq \Lambda\tilde{f} \in \{1\}^\perp$, i.e. Λf is not a multiple of the identity.
If f would be polynomially bounded with respect to T, then by Remark 3.4 also Λf is polynomially bounded with respect to $\hat{T}$.
Because Λf is not a multiple of the identity and $\hat{T}$ is a weakly mixing automorphism, we conclude on the other hand by Theorem 3.3 that Λf is not polynomially bounded with respect to $\hat{T}$. To avoid a contradiction we have to admit that f is not polynomially bounded with respect to T. This proves Theorem 3.4.

Remark 3.5: In [10] it is also shown that for a transition operator T we have $\lim_{n\to\infty} \|T^n f\|_2 = 0$ for all $f \in \{1\}^\perp$ if and only if a corresponding minimal Markov dilation is a K–system in the sense of ergodic theory. The transition operator T is called completely mixing in this case. This establishes a nontrivial range of applications for Theorem 3.4. An explicit construction of vectors for which Theorem 3.4 is applicable can be found in [12, p.113ff.].

If we try to find generalizations of these results for more general C^*–algebras (as we succeeded in Theorem 3.2 for Theorem 3.1), there are difficulties arising from the complexity of noncommutative ergodic theory. In particular the author has not been able to give a necessary and sufficient condition for endomorphisms to fulfil $X_{pb} = \mathbb{C}1$ as we have succeeded to give for commutative algebras in Theorem 3.3. We can show however that this phenomenon also exists in the noncommutative setting if we assume very strong mixing properties. This is the intention of the following considerations.

Let $\mathcal{A}$ be a C^*–algebra with a unit and $\hat{\mathcal{A}} := \bigotimes_{n\in\mathbb{N}_0} \mathcal{A}$ the infinite minimal C^*–tensor product.
The tensor shift $T : \hat{\mathcal{A}} \mapsto \hat{\mathcal{A}}$ is the C^*–endomorphism which is defined by $T(\bigotimes_{n=0}^{N} 1 \otimes x \otimes 1 \otimes 1 \otimes 1 \ldots) = \bigotimes_{n=0}^{N+1} 1 \otimes x \otimes 1 \otimes 1 \ldots)$ for all N and all $x \in \mathcal{A}$.

Theorem 3.5 *Let $\mathcal{A}$ be a C^*–algebra with a unit, $\hat{\mathcal{A}} := \bigotimes_{n\in\mathbb{N}_0}\mathcal{A}$ and T the tensor shift. Every vector which is polynomially bounded with respect to T is a multiple of the identity, i.e. $\hat{\mathcal{A}}_{pb} = \mathbb{C}1$.*

In the proof of Theorem 3.5 we shall need the following modification of the Rosenthal-Dor lemma:

Lemma: Let $\mathbf{H}$ be a Hilbert space, $\{q_n\}_{n=1}^{2^{J+2}-1} \subset \mathcal{B}(\mathbf{H})$ a family of nonvanishing projections with $q_{2n} \le q_n, q_{2n+1} \le q_n, q_{2n}q_{2n+1} = 0$ for all
$1 \le n \le 2^{J+1} - 1$.
Let $\{x_j\}_{j=0}^{J} \subset \mathcal{B}(\mathbf{H})$ be a sequence of normal operators, $D_0, D_1 \subset \mathbb{C}$ disks with diameter d and distance δ, $d \le \frac{1}{2}\delta$, furthermore $p_{D_0}(x_j)$ and $p_{D_1}(x_j)$ the corresponding spectral projections of x_j,

$$E_j := \bigvee_{\substack{n=2^{j+1}\\ n\,even}}^{2^{j+2}-1} q_n,\ O_j := \bigvee_{\substack{n=2^{j+1}\\ n\,odd}}^{2^{j+2}-1} q_n,\ j = 0, \dots, J.$$

If $E_j \le p_{D_0}(x_j)$ and $O_j \le p_{D_1}(x_j)$, $j = 0, \dots, J$ then we have

$$\frac{\delta}{8}\sum_{j=0}^{J}|\gamma_j| \le \|\sum_{j=0}^{J}\gamma_j x_j\|$$

for all $(\gamma_0, \gamma_1, \dots, \gamma_J) \in \mathbb{C}^{J+1}$.

Note that the projections $\{q_n\}_{n=1}^{2^{J+2}-1}$ necessarily commute, while this need not be the case for the normal elements $\{x_j\}_{j=0}^{J}$ or their spectral projections.

Proof of the lemma: To apply the original Rosenthal–Dor lemma we give a functional representation of the operators on the unit sphere $\mathbf{H}_1$ of the Hilbert space $\mathbf{H}$: For any $x \in \mathcal{B}(\mathbf{H})$ let $\hat{x}$ be the function

$$\hat{x} : \mathbf{H}_1 \to \mathbb{C},\ \hat{x}(h) :=< xh, h > .$$

We have $\|x\| = \|\hat{x}\|_\infty$.

To every projection p we can associate the image
$p(\mathbf{H}_1) = \{h \in \mathbf{H}_1 : ph = h\} = \{h \in \mathbf{H}_1 : \hat{p}(h) =< ph, h >= 1\}$.
From the assumptions we infer

$$\bigcup_{\substack{n=2^{j+1}\\ n\,even}}^{2^{j+2}-1} q_n(\mathbf{H}_1) \subset E_j(\mathbf{H}_1),\ \bigcup_{\substack{n=2^{j+1}\\ n\,odd}}^{2^{j+2}-1} q_n(\mathbf{H}_1) \subset O_j(\mathbf{H}_1),\ j = 0, \dots, J.$$

Suppose $h \in E_j(\mathbf{H}_1) \subset p_{D_0}(x_j)(\mathbf{H}_1)$.

Then $\hat{x}_j(h) =< x_j h, h >=< x_j p_{D_0}(x_j) h, h >$. This is an element of the convex hull of the spectrum of $x_j p_{D_0}(x_j)$ and therefore belongs to D_0. In an analogous way we infer $\hat{x}_j(h) \subset D_1$ if $h \in O_j(\mathbf{H}_1)$. We can now apply the Rosenthal–Dor lemma (respectively Remark 3.3(a)) and obtain

$$\|\sum_{j=0}^{J} \gamma_j x_j\| = \|\sum_{j=0}^{J} \gamma_j \hat{x}_j\|_\infty \geq \frac{\delta}{8} \sum_{j=0}^{J} |\gamma_j|$$

for all $(\gamma_0, \gamma_1, \ldots, \gamma_J) \in \mathbb{C}^{J+1}$.

Proof of Theorem 3.5: We shall proceed in several steps. Some results are interesting in their own right. It is convenient to give the following definition:

Definition 3.1 *Let X be a Banach space, $T : X \to X$ any map.*
For any $x \in X$ we define the transition set trans(x) (with respect to T) to be the following subset of $\mathbb{N}$:
trans(x) :=$\{n \in \mathbb{N} : \{T^{nj}x\}_{j \in \mathbb{N}_0}$ is equivalent to the unit vector base of $l^1(\mathbb{N}_0)\}$.

Obviously a vector x with a nonempty transition set trans(x) cannot be polynomially bounded, and the transition set may be regarded as a coarse measure of deviation from polynomial boundedness. We shall give a more detailed discussion of this in the fourth section.

We return now to the proof of Theorem 3.5:
Suppose $\mathcal{A}_0 := \mathcal{A} \otimes 1 \subset \hat{\mathcal{A}}$ (i.e. $\mathcal{A}$ at position zero in the tensor product),
$\mathcal{A}_n := T^n \mathcal{A}_0$, $\mathcal{A}_{[m,n]} := \bigvee_{k=m}^{n} \mathcal{A}_k$ (C^*–algebraic hull).
For any element x of a C^*–algebra we denote by $\delta(x)$ the diameter of its spectrum, i.e. $\delta(x) := \max\{|\lambda - \mu| : \lambda, \mu \in sp(x)\}$.

(a) If $x \in \mathcal{A}_{[0,L-1]} \setminus \mathbb{C}1$ is normal then $[L, \infty) \subset trans(x)$. We can use a common constant c $= \frac{\delta(x)}{8}$.

Proof: To apply the lemma we start with a faithful representation $(\pi, \mathcal{H})$ of $\mathcal{A}_{[0,L-1]}$. By forming suitable finite tensor products we are able to represent products of spectral projections of x and of its translates by T^L as elementary tensors.

It is possible to choose disks $D_0, D_1 \subset \mathbb{C}$ with diameter d and distance δ, $d \leq \frac{1}{2}\delta$, so that the spectral projections $p_i = p_{D_i}(x), i = 0, 1$, do not vanish.

Indeed for any $\delta' > 0$ we can choose $\delta > \delta(x) - \delta'$.

We are now able to define a tree $\{q_n\}$ as it is used in the lemma:

$q_1 := 1,$

$q_2 := p_0,\ q_3 := p_1$

$q_4 := p_0 \cdot T^L p_0,\ q_5 := p_0 \cdot T^L p_1,\ q_6 := p_1 \cdot T^L p_0,\ q_7 := p_1 \cdot T^L p_1, \ldots$ etc.,

in general:

$q_n := p_{\epsilon_j} \cdot T^L p_{\epsilon_{j-1}} \cdot T^{2L} p_{\epsilon_{j-2}} \cdot \ldots \cdot T^{jL} p_{\epsilon_0}$ if $n = 2^{j+1} + 2^j \epsilon_j + 2^{j-1}\epsilon_{j-1} + \ldots + \epsilon_0$.

We may also write

$q_n = p_{\epsilon_j} \otimes p_{\epsilon_{j-1}} \otimes p_{\epsilon_{j-2}} \otimes \ldots \otimes p_{\epsilon_0} \in \pi(\mathcal{A}_{[0,L-1]})'' \otimes \pi(\mathcal{A}_{[L,2L-1]})'' \otimes \ldots$
$\ldots \otimes \pi(\mathcal{A}_{[jL,(j+1)L-1]})''$.

This representation shows clearly that the assumptions of the lemma concerning $\{q_n\}$ are valid here. In particular all q_n are nonvanishing. We also have

$$E_j := \bigvee_{n=2^{j+1}, n\, even}^{2^{j+2}-1} q_n \leq T^{jL} p_0 = p_{D_0}(T^{jL}x)$$

$$O_j := \bigvee_{i=2^{j+1}, n\, odd}^{2^{j+2}-1} q_n \leq T^{jL} p_1 = p_{D_1}(T^{jL}x),$$

Applying the lemma we conclude

$$\frac{\delta}{8} \sum_{j=0}^{J} |\gamma_j| \leq \| \sum_{j=0}^{J} \gamma_j T^{Lj} x \|$$

for all $(\gamma_0, \ldots, \gamma_J) \in \mathbb{C}^{J+1}$ and $J \in \mathbb{N}_0$. The same argument is also possible for $L+1, L+2, \ldots$ instead of L. This proves (a).

(b) If for $x \in \hat{\mathcal{A}}$ there is a normal element $y \in \mathcal{A}_{[0,L-1]}$ with $\|x - y\| \leq \epsilon$, then we have for all $l \geq L$:

$$(\frac{\delta(y)}{8} - \epsilon) \sum_{j=0}^{J} |\gamma_j| \leq \| \sum_{j=0}^{J} \gamma_j T^{lj} x \|$$

for all $(\gamma_0, \ldots, \gamma_J) \in \mathbb{C}^{J+1}$ and $J \in \mathbb{N}_0$.

Proof: By (a) we get for the normal element y

$$\frac{\delta(y)}{8} \sum_{j=0}^{J} |\gamma_j| \leq \| \sum_{j=0}^{J} \gamma_j T^{lj} y \|$$

for all $l \geq L$, $(\gamma_0, \ldots, \gamma_J) \in \mathbb{C}^{J+1}$ and $J \in \mathbb{N}_0$. Because of $\|x - y\| \leq \epsilon$ we infer

$$\| \sum_{j=0}^{J} \gamma_j T^{lj} x \| \geq \| \sum_{j=0}^{J} \gamma_j T^{lj} y \| - \| \sum_{j=0}^{J} \gamma_j T^{lj} (x - y) \|$$

$$\geq \frac{\delta(y)}{8}\sum_{j=0}^{J}|\gamma_j| - \|x-y\|\sum_{j=0}^{J}|\gamma_j| \geq (\frac{\delta(y)}{8}-\epsilon)\sum_{j=0}^{J}|\gamma_j|$$

This proves (b).

(c) If x,y are selfadjoint then $|\delta(x)-\delta(y)| \leq 2\|x-y\|$.

Proof: Choose $0 < \lambda \in \mathbf{R}$ large enough so that

$x+\lambda 1 > 0,\ y+\lambda 1 > 0,\ x-\lambda 1 < 0,\ y-\lambda 1 < 0.$

For selfadjoint elements norm and spectral radius are equal. We infer

$\delta(x) = \max sp(x) - \min sp(x) = (\|x+\lambda 1\| - \lambda) - (-\|x-\lambda 1\| + \lambda)$

$= \|x+\lambda 1\| + \|x-\lambda 1\| - 2\lambda.$

Analogous for y.

Therefore we have

$|\delta(x)-\delta(y)| = |(\|x+\lambda 1\| + \|x-\lambda 1\| - 2\lambda) - (\|y+\lambda 1\| + \|y-\lambda 1\| - 2\lambda)|$

$\leq \|(x+\lambda 1)-(y+\lambda 1)\| + \|(x-\lambda 1)-(y-\lambda 1)\| = 2\|x-y\|.$

This proves (c).

(d) If $x \in \hat{\mathcal{A}} \setminus \mathbb{C}1$ is selfadjoint then $\mathbb{N} \setminus trans(x)$ is finite. More precise: If we choose $L \in \mathbb{N}$ minimal with the property that there is an element $y \in \mathcal{A}_{[0,L-1]}$ with $\|x-y\| < \frac{\delta(x)}{10}$ then $[L,\infty) \subset trans(x)$.

Proof: If $x \in \hat{\mathcal{A}} \setminus \mathbb{C}1$ is selfadjoint then we have $\delta(x) > 0$. Because $\bigcup_{L\in\mathbb{N}} \mathcal{A}_{[0,L-1]}$ is (norm-)dense in $\hat{\mathcal{A}}$ we can find a $L \in \mathbb{N}$ minimal with the property that there is an element $y \in \mathcal{A}_{[0,L-1]}$ with $\|x-y\| < \frac{\delta(x)}{10}$. We may assume that y is also selfadjoint: otherwise replace y by $\frac{1}{2}(y+y^*)$.

Using (c), an easy computation gives

$\|x-y\| < \frac{1}{8}(\delta(x) - 2\|x-y\|) \leq \frac{\delta(y)}{8}$. Now (d) follows by an application of (b).

(e) $\hat{\mathcal{A}}_{pb} = \mathbb{C}1$.

Proof: By (d) we already know that a selfadjoint element which is not a multiple of the identity is not polynomially bounded. Suppose there is any $x \in \hat{\mathcal{A}} \setminus \mathbb{C}1$ which is polynomially bounded with respect to T. Then x* is also polynomially bounded:

$\|p(T)x^*\| = \|\sum_{j=0}^{J}\gamma_j T^j x^*\| = \|\sum_{j=0}^{J}(\overline{\gamma_j}T^j x)^*\| = \|\sum_{j=0}^{J}\overline{\gamma_j}T^j x\|$

$\leq K_x\|q\|_\infty$ with $q(z) = \sum_{j=0}^{J}\overline{\gamma_j}z^j$. Obviously $\|p\|_\infty = \|q\|_\infty$.

We conclude that the selfadjoint elements $x+x^*$ and $i(x-x^*)$ are also polynomially bounded, and the decomposition $x = \frac{1}{2}[(x+x^*) - i(i(x-x^*))]$

shows that at least one of them is not a multiple of the identity. This is a contradiction to the assertions established above.
Theorem 3.5 is now completely demonstrated.

Remark 3.6: If we specialize $\mathcal{A} := \mathbb{C}^n$ then Theorem 3.5 deals with the symbolic shift with n symbols. This shift is weakly mixing for suitable measures (for example product measures) and we could have achieved the result of Theorem 3.5 in this special case by using Theorem 3.3. But the proof of Theorem 3.5 shows that the stronger mixing properties of a symbolic shift allow to construct a lot of topological embeddings of the sequence space $l^1(\mathbb{N}_0)$ into the orbits of the shift, while in the proof of Theorem 3.3 we only worked with (arbitrarily long) finite sections. These embeddings correspond to what we have called nonempty transition sets. In regarding the symbolic shift or the easy generalization of the result to (irreducible and aperiodic) subshifts of finite type (see [11]) we can indeed notice a natural relation between transition sets and transitions for shifts or subshifts in the usual sense. This motivates our terminology.

4 The Transition Set

Let T be a contraction on a Banach space X and $x \in X$. Recall the definition of the transition set: A number $n \in \mathbb{N}$ belongs to trans(x) if and only if $\{T^{nj}x\}_{j\in\mathbb{N}_0}$ is equivalent to the unit vector base of $l^1(\mathbb{N}_0)$. Using this concept we shall define a similarity invariant for T-invariant subspaces, and by actually computing this invariant in concrete examples we can perform some classification.
We first extend the definition above:

Definition 4.1 *Let T be a contraction on a Banach space X.*
For any subset $Y \subset X$ we define the transition set trans(Y) to be the union of all trans(y) with $y \in Y$.
In particular for any $x \in X$ we can speak of the transition set of the cyclic subspace C(x) generated by x (i.e. C(x) := $\overline{lin}\{T^n x\}_{n\in\mathbb{N}_0}$).

Let us first state some useful properties of transition sets:

Lemma: Let T be a contraction on a Banach space X.
(a) If $x \in X$ and $S : C(x) \to X$ is a bounded linear operator which commutes with T, then $trans(Sx) \subset trans(x)$.
(b) Let Y be any subset of X. Then $trans(\partial Y) \subset trans(Y) = trans(\overline{Y})$.

Proof: Ad (a): Because of $trans(0) = \emptyset$ we may assume $S \neq 0$. If we have $n \in trans(Sx)$ then there is a constant $c > 0$ so that for any polynomial p we get $c\|p\|_A \leq \|p(T^n)Sx\| = \|Sp(T^n)x\| \leq \|S\|\,\|p(T^n)x\|$
or equivalently $\|p(T^n)x\| \geq c\|S\|^{-1}\|p\|_A$ which implies $n \in trans(x)$.
Ad (b): Suppose $n \in trans(\partial Y)$. There is an element $x \in \partial Y$ such that $n \in trans(x)$, i.e. there is a constant $c > 0$ so that for all polynomials p we have $c\|p\|_A \leq \|p(T^n)x\|$. Choose $y \in Y$ with $\|x - y\| \leq \frac{c}{2}$. Then we find
$\|p(T^n)y\| \geq \|p(T^n)x\| - \|p(T^n)(x - y)\| \geq c\|p\|_A - \frac{c}{2}\|p\|_A = \frac{c}{2}\|p\|_A$.
We infer $n \in trans(y)$ and therefore $n \in trans(Y)$. Because the boundary set ∂Y adds nothing to the transition set of Y, the equation
$trans(Y) = trans(\overline{Y})$ is an immediate consequence.

We can now prove the announced relation between similarity and transition sets. As an abbreviation let us call two T-invariant subspaces of an operator T similar if the respective restrictions of T to these subspaces are similar.

Theorem 4.1 *Let X be a Banach space, $T : X \to X$ a contraction on X.*
(a) If C_1, C_2 are similar subspaces of T then $trans(C_1) = trans(C_2)$,
i.e. the transition set is a similarity invariant for T-invariant subspaces.
(b) For all $x \in X$ we have $trans(x) = trans(C(x))$,
i.e. the transition set of a cyclic subspace can be calculated as the transition set of any cyclic vector.

Proof: Ad (a): If $\Lambda : C_1 \to C_2$ is a similarity of $T|_{C_1}$ and $T|_{C_2}$, then apply part (a) of the lemma for all elements of C_1 and C_2 with $S = \Lambda$ respectively $S = \Lambda^{-1}$. We infer $trans(x) = trans(\Lambda x)$ for all $x \in C_1$, and the assertion follows because Λ is bijective.
Ad (b): Suppose $y \in$C(x). If there is a polynomial p with y=p(T)x, then

by part (a) of the lemma with S := p(T) we infer $trans(y) \subset trans(x)$. Any element $y \in$C(x) can be approximated by elements of this special form. Using part (b) of the lemma we see that the inclusion $trans(y) \subset trans(x)$ remains valid also in the general case. This proves (b).

Remark 4.1: If we have a bounded inverse T^{-1} of the operator T in Theorem 4.1, then (b) remains valid for a two-sided cyclic subspace $\overline{lin}\{T^n x\}_{n\in\mathbf{Z}}$ and its (two-sided) cyclic vector x. This follows by an easy modification of our proof above: use polynomials in T and T^{-1}.

We now turn to methods which allow us to compute transition sets. Some notation for certain subsets of ∂D will be useful: For any $n \in \mathbb{N}$ and $\alpha \in [0,1)$ define $\Delta_n^\alpha := \{\exp\left[2\pi i(\frac{k}{n}+\alpha)\right] : k = 0,\dots,n-1\}$. Geometrically Δ_n^α is the set of vertices of a regular n-gon.

Lemma: Let T be a contraction on a Banach space X and $x \in X$. If $\alpha \in [0,1)$ and if p is a polynomial with $p|_{\Delta_n^\alpha} = 0$ then $n \notin trans(p(T)x)$.

Proof: We first assume $\alpha = 0$.

Because of its zero set we conclude that p contains a factor q with $q(z) = z^n - 1$. By part (a) of the lemma above it is enough to show that $n \notin trans(q(T)x)$, and to prove this we show that $\{T^{nj}q(T)x\}_{j\in\mathbb{N}_0}$ is not equivalent to the unit vector base of $l^1(\mathbb{N}_0)$ (for all $x \in X$).

For any $(\gamma_0,\dots,\gamma_J) \in \mathbb{C}^{J+1}$ we get: $\sum_{j=0}^J \gamma_j T^{nj} q(T)x =$

$$= \sum_{j=0}^J \gamma_j T^{nj}(T^n x - x) = (-\gamma_0 x) + (\gamma_0-\gamma_1)T^n x + \dots + (\gamma_{J-1}-\gamma_J)T^{Jn}x + \gamma_J T^{(J+1)n}x.$$

Choose $\gamma_j := \frac{1}{j}$ if $j \neq 0$ (and 0 otherwise).

Setting $p_J(z) := \sum_{j=0}^J \gamma_j z^j = \sum_{j=0}^J \frac{1}{j} z^j$ we get

$$\|p_J(T^n)q(T)x\| = \|-T^n x + (1-\frac{1}{2})T^{2n}x + \dots + [\frac{1}{J-1} - \frac{1}{J}]\, T^{Jn}x + \frac{1}{J}T^{(J+1)n}x\|$$

$$\leq (1 + \sum_{j=1}^{J-1}[\frac{1}{j} - \frac{1}{j+1}] + \frac{1}{J})\|x\| = 2\|x\|.$$

This is bounded if $J \to \infty$, but

$$\lim_{J\to\infty} \|p_J\|_A = \lim_{J\to\infty} \sum_{j=0}^J \frac{1}{j} = \infty.$$

The assertion is now proved for $\alpha = 0$. For $\alpha \in [0,1)$ we can use the same proof with some modifications: $q(z) := z^n - e^{2\pi i\alpha n}, \gamma_j := \frac{1}{j}e^{-2\pi i\alpha nj}$ if $j \neq 0$.

The lemma shows that sets of zeros of the form Δ_n^α are obstructions for the natural number n to belong to certain transition sets, independent of the operator T or the vector x under discussion. The following detailed determination of transition sets for a concrete operator shows that it is possible that there are no other obstructions.

Consider the shift operator S on $l^1(\mathbf{Z})$ defined by $(Sx)_n := x_{n-1}$ for all $x = \{x_n\}_{n\in\mathbf{Z}} \in l^1(\mathbf{Z})$. Equivalently (via Fourier transform) we may consider the multiplication operator M_z on the Wiener algebra $A(\partial D)$.

Theorem 4.2 *Suppose* $X = A(\partial D)$, $T := M_z$ *and* $f \in X$.
For any $n \in \mathbb{N}$ *the following assertions are equivalent:*
(1) $n \notin trans(f)$.
(2) There is a number $\alpha \in [0,1)$ *such that* $f|_{\Delta_n^\alpha} = 0$.

Proof:

$$(2) \Rightarrow (1)$$

This is proved above for polynomials. In the general case we use the following approximation argument:

We shall restrict ourselves again to the case $\alpha = 0$ and leave the modifications necessary in the other cases to the reader.

The set Δ_n^0 is finite and therefore a set of spectral synthesis (see [9, Chap.V.1]) and we conclude that there is only one closed ideal in $A(\partial D)$ whose zero set is exactly Δ_n^0. In particular the ideal of all functions vanishing in Δ_n^0 (which includes f) coincides with the closed principal ideal generated by the polynomial q with $q(z) = z^n - 1$. Thus there is a sequence $\{g_k\}_{k\in\mathbb{N}} \subset A(\partial D)$ such that $\lim_{k\to\infty} \|f - g_k q\|_A = 0$.

Successive application of the lemmata yields
$n \notin trans(q), n \notin trans(g_k q)$ and $n \notin trans(f)$.

$$(1) \Rightarrow (2)$$

Suppose that Δ_n^α is not contained in the zero set of f for any $\alpha \in [0,1)$. To any $z = z_1 \in \partial D$ we can associate the numbers $\{z_k\}_{k=1}^n$ with the same n-th power, i.e. $z_1^n = z_2^n = \ldots = z_n^n$. There is at least one $k \in \{1, \ldots, n\}$ such that $f(z_k) \neq 0$. We call this a suitable choice of k.

We consider an operator $\Gamma : A(\partial D) \to A(\partial D), (\Gamma g)(z) := g(z^n)f(z)$. Obviously Γ is bounded. It is also injective: If $g \not\equiv 0$ then $g(z^n) \neq 0$ for some z and for a suitable $k \in \{1, \ldots, n\}$ we have $(\Gamma g)(z_k) = g(z^n)f(z_k) \neq 0$.

The range of Γ is closed in $A(\partial D)$:

Suppose $h \in A(\partial D)$ to be in the closure of $\Gamma(A(\partial D))$. We can choose a sequence $\{h_j\}_{j\in\mathbb{N}} \subset \Gamma(A(\partial D))$ with $\lim_{j\to\infty} \|h - h_j\|_A = 0$. There are functions $\{g_j\}_{j\in\mathbb{N}} \subset A(\partial D)$ such that $h_j(z) = g_j(z^n)f(z)$ for all $z \in \partial D$.

We have $g_j(z^n) = \frac{h_j(z_k)}{f(z_k)}$ for any suitable k. Because of $\|h - h_j\|_A \to 0$ we also have $\|h - h_j\|_\infty \to 0$, and thus it is possible to define a continuous function g on ∂D such that for all $z \in \partial D$ (and corresponding suitable k):

$$g(z^n) = \lim_{j\to\infty} g_j(z^n) = \lim_{j\to\infty} \frac{h_j(z_k)}{f(z_k)} = \frac{h(z_k)}{f(z_k)}$$

or equivalently $h(z_k) = g(z^n)f(z_k)$.

The equation $h(z) = g(z^n)f(z)$ remains also true if f(z)=0 because of $h(z) = \lim_{j\to\infty} h_j(z) = \lim_{j\to\infty} g_j(z^n)f(z) = 0$.

Thus it only remains to show that $g \in A(\partial D)$: For any $z_1 \in \partial D$ we have $g(z_1^n) = \frac{h(z_k)}{f(z_k)}$ for a suitable k. There is a neighbourhood U of z_1 such that $f|_{z_k z_1^{-1} U}$ has no zero value. We can extend $f|_{z_k z_1^{-1} U}$ to a function $\tilde{f} \in A(\partial D)$ with no zero values on ∂D: for example we can take $\tilde{f} = f + \psi$ where ψ is any C^1–function on ∂D which is zero on $z_k z_1^{-1} U$ and differs from -f on $\partial D \setminus z_k z_1^{-1} U$.

By a theorem of Wiener (see [9, Chap.V.2]) $\tilde{f}$ is an invertible element of $A(\partial D)$. We have $g(z^n) = h\tilde{f}^{-1}(z_k z_1^{-1} z)$ for all $z \in U$.

We have proved that the function $z \mapsto g(z^n)$ belongs locally to $A(\partial D)$, i.e. for every $z \in \partial D$ there is a neighbourhood of z inside of which the function coincides with the restriction of a function in $A(\partial D)$ to this neighbourhood. By another theorem of Wiener (see [9, Chap.II.4]) this implies that the function $z \mapsto g(z^n)$ actually belongs to $A(\partial D)$. But then we have also $g \in A(\partial D)$, and indeed the range of Γ is closed.

We can now finish the proof of Theorem 4.2. We conclude from the considerations above that the operator Γ has a bounded inverse, i.e. there is a constant $c > 0$ such that $\|\Gamma g\|_A \geq c\|g\|_A$ for all $g \in A(\partial D)$. In particular for any polynomial p=g we have $\|p(T^n)f\|_A = \|\Gamma p\|_A \geq c\|p\|_A$. This shows that $n \in trans(f)$.

Proposition: Suppose $X := A(\partial D)$, $T := M_z$.
For a subset $A \subset \mathbb{N}$ the following assertions are equivalent:
(1) There is a function $f \in X, f \not\equiv 0$, such that $A = \mathbb{N} \setminus trans(f)$.
(2) A is finite and contains all divisors of its elements.

Proof: For (1) $\Rightarrow$ (2) observe that if $A = \mathbb{N} \setminus trans(f)$ is infinite then the zero set of f contains sets of the form Δ_n^α with arbitrarily large n. Thus the zero set is dense in ∂D and by the continuity of f we infer $f \equiv 0$.
The condition concerning divisors is trivial for cosets of transition sets with respect to $\mathbb{N}$ in general.

To prove (2) $\Rightarrow$ (1) define a polynomial with a suitable zero set such that $A = \mathbb{N} \setminus trans(p)$.

Proposition: Suppose $X := A(\partial D)$, $T := M_z$.
There are infinitely many similarity classes of cyclic subspaces (one-sided or two-sided). The only two-sided cyclic subspace similar to X is X itself.

Note that in this case a two-sided cyclic subspace is the same as a closed principal ideal of the algebra $A(\partial D)$ (see [4, 11.1]). It is not known to the author if the problem of similarity of such ideals has been considered before.

Proof: By the proposition above there are infinitely many elements of $A(\partial D)$ with pairwise different transition sets. Using Theorem 4.1 (respectively Remark 4.1) we conclude that the cyclic subspaces generated by these elements belong to pairwise different similarity classes. If a cyclic subspace is similar to $A(\partial D)$ then its transition set is $\mathbb{N}$. We conclude that any cyclic function has no zeros on ∂D and is thus invertible in $A(\partial D)$ by Wiener's theorem. Therefore the ideal generated by it is $A(\partial D)$ itself.

Remark 4.2: It is easy to check that the assertions in Theorem 4.2 and the propositions following it remain valid if we consider $T = M_z$ on $A^+(\partial D)$. Of course in the last proposition we can here only consider one-sided cyclic subspaces which are in this case the same as closed principal ideals of the algebra $A^+(\partial D)$. The result that there are infinitely many different similarity classes of cyclic subspaces for the one-sided shift on $l^1(\mathbb{N}_0)$ should be compared to the totally different situation for the one-sided shift on $l^2(\mathbb{N}_0)$, where Beurling's theorem states (among other things) that the restriction of the shift to cyclic subspaces is always unitarily equivalent (and therefore also similar) to the shift itself (see [7]).

For the shift on $l^1(\mathbb{N}_0)$ we can give the following characterization of the cyclic subspaces which are similar to $l^1(\mathbb{N}_0)$ itself:

Proposition: Suppose $X := A^+(\partial D)$, $T := M_z$, $f \in X$.
The following assertions are equivalent:
(1) C(f) is similar to C(1)=X.
(2) trans(f) = $\mathbb{N}$.
(3) $f \neq 0$ everywhere on ∂D.
(4) C(f) = X or C(f) = $(z - z_1)^{n_1}(z - z_2)^{n_2} \ldots (z - z_k)^{n_k} \cdot A^+(\partial D)$, where $z_1, \ldots, z_k$ are finitely many complex numbers in D and $n_1, \ldots, n_k \in \mathbb{N}$.

Proof: (1) $\Rightarrow$ (2) follows from Theorem 4.1 and (2) $\Rightarrow$ (1) is immediate from the definition of the transition set.
(2) $\Leftrightarrow$ (3) follows from Theorem 4.2 and Remark 4.2.
The equivalence of (3) and (4) is more or less implicit in [9, Chap.XI.3]. We give the main arguments: The function f being analytic in D and nonvanishing on ∂D by (3), can only have finitely many zeros in D. If there is no zero at all then f is invertible (by Gelfand theory for the Banach algebra $A^+(\partial D)$).

So assume we have zeros $z_1, \ldots, z_k$. Let n_1 be the maximal number such that all functions in C(f) contain the factor $(z - z_1)^{n_1}$. Then the set of all functions $g \in A^+(\partial D)$ with the property that the function $z \mapsto (z-z_1)^{n_1} g(z)$

belongs to C(f) is an ideal whose set of zeros does not contain z_1. Iterating this procedure we obtain the representation (4).

Remark 4.3: If $trans(f) \neq \mathbb{N}$ the classification of the corresponding similarity classes of cyclic subspaces remains an open problem.

Remark 4.4: Some of the results above can be transferred to other operators: If T is any contraction on a Banach space X and for $x \in X$ we have $trans(x) \neq \emptyset$ then in the cyclic subspace C(x) there are cyclic subspaces belonging to infinitely many different similarity classes. For a proof observe that if $n \in trans(x)$ then for any $f \in A^+(\partial D)$ we have $jn \in trans(f(T^n)x)$ if and only if $j \in trans(f)$ with respect to M_z in $A^+(\partial D)$ ($j \in \mathbb{N}$).

For example this remark is applicable for the tensor shift analyzed in Theorem 3.5.

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