Quantum Control: Approach based on Scattering Theory for Non-commutative Markov Chains and Multivariate Operator Theory
Part 2 - Proposed Research and its Context

The aim of this project is to apply mathematical techniques in Scattering Theory for Noncommutative Markov Chains, and Multivariate Operator Theory to problems in the rapidly developing interdisciplinary field of Quantum Control. The proposal draws on the expertise of the Quantum Control and Information research group at Aberystwyth, and will include research collaboration with specialists Burkhard Kümmerer (Darmstadt) and Hans Maassen (Nijmegen) who appear in the proposal as visiting researchers. The project aims to develop connections between these fields, and to combine them in a new way toward developing a systematic theory of quantum control.

1 Introduction

Modern control theory has frequently used concepts and results from abstract mathematics. The aim of this proposal is to explore genuinely noncommutative versions with a view toward direct applications to the emergent discipline of quantum control. Experimental advances mean that physicists have an unprecedented ability to manipulate quantum mechanical systems, and from the technological point of view there is currently much interest in deriving a theory of quantum engineering as the foundation for a much anticipated quantum technological revolution, see for instance the well-known article by Dowling and Milburn [DowlMil].

Hideo Mabuchi, one of the foremost experimentalists and theoreticians in quantum control theory, recently wrote about the core problem facing the subject [Ma]: '*Substantial progress has been made over the past two decades in the development of intuitive approaches within specific application areas, but the formulation of an integrated, first-principles discipline of quantum control - as a proper extension of classical control theory - remains a broad priority.*' The current proposal wishes to contribute to this goal by providing an integrated approach based on the inclusion and adaptation of recent mathematical research in operator theory.

2 Proposed Research and its Context

Control Theory. The theory of control has many deep and productive connections with disciplines of mathematical analysis. In particular, the state space models of control theory [Ki,Za], i.e. the description of the system by an internal state space with input/output channels, can be formulated using the language of operator theory. In a seminal paper of Helson [He], classical state space models in discrete time were given an operator theoretic description and then shown to be closely related to the Nagy-Foias theory of unitary dilations for contractions [NF]. In particular, the transfer function of a system was shown to correspond to the characteristic function of a contraction. This moreover provides a natural way to address issues of controllability and observability. Looking at the input/output formalism from the point of view of Lax-Phillips scattering theory [LP] , the transfer function evidently contains information equivalent to the wave operators.

Another related contact between control and operator theory has been developed in the area of robust control. The concept of robustness has been one of the main areas of investigation in modern control engineering [Ki] and there are important relations between H^∞-control and the operator theoretic version of Nevanlinna-Pick interpolation, in particular when combined with Schur-type algorithms to describe cascades of systems. Again this fits in with the modern view point that control is essentially interconnection of systems. See Foias and Frazho [FF] for an overview from the point of view of operator theory.

Quantum Control. Recent experimental progress in applying feedback control to individual quantum systems has given a timely impetus for the field of quantum filtering and control. This work requires a substantial extension of classical control theory due to the subtleties of quantum systems and promises many novel applications [DowlMil]. With the tendency toward miniaturization and nanotechnology, quantum control is certain to be an essential element in future quantum technologies. The more realistic situation of controlling quantum gate operators in finite time, for instance, will have important implications for practical quantum information processing. See [BHJ1,BHJ2] for an overview.

It is possible to develop the theoretical work on the basis of pre-existing knowledge on open quantum systems, as many of the mathematically rigorous formulations and results already exist, particularly in the quantum probability literature. Here there is a well developed general theory of the dynamical behaviour of open quantum systems generalizing Markov models in classical probability theory to a non-commutative setting, for instance, see the book of Parthasarathy [Pa]. Moreover the theory of continuous-time quantum state estimation (filtering,[Be]) has been designed for precisely the problems now being raised by physicists.

However it is also quite natural to identify an open quantum system with the state space of control theory and to treat its interactions with its surroundings as input and output channels. Pioneering work about filtering based on quantum measurements has been done from this point of view by Belavkin [Be] and this is now integrated into a theory of quantum feedback. See [BHJ1,BHJ2]. Recently coherent feedback (without measurement) has received theoretical attention [JNP] as well as an analysis of experimental feasibility [Ma].

In [GJ1,GJ2] Gough (co-investigator in this proposal) and James have developed a theory of quantum feedback networks extending classical stochastic control networks. Inputs and outputs are bosonic fields on symmetric Fock space and the input/output formalism is modeled by means of the Hudson-Parthasarathy quantum stochastic calculus [Pa]. Their theory is in fact genuinely nonlinear, but for the restriction to linear systems (local oscillators with feedforward and beam-splitter feedback) one may treat the models using transfer function techniques based on Laplace transforms of the noncommutative fields [GGY]. This in turn is a generalization of the work of Yanagisawa and Kimura [YK]. In figure 1 we show an example of the type of setup which the new theory can model: rather than driving the quantum system (the plant) using a channel input and applying some direct feedback (either based on measurement of the output, or fully quantum), we instead use a beam splitter to establish a topologically nontrivial feedback loop. The plant is now in loop and its characteristics as a dynamical system will of course be modified. For a linear plant in-loop, the modified characteristic function will be a Möbius transform of the original.

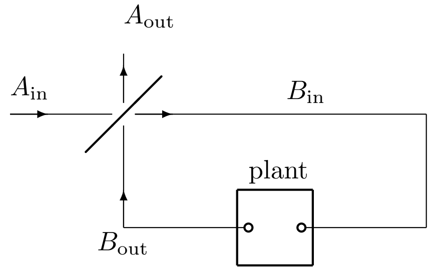

Figure 1: Feedback using a beam-splitter.

Multivariate Operator Theory. Multivariate operator theory [Ar,AV] provides extensions of many of the operator theoretic results relevant to classical control theory. Instead of considering a single operator now the joint action of a tuple of operators, in general not commuting with each other, is investigated. While the single operator case provides hints what to look for, the multivariate setting gives rise to many new and subtle phenomena. See for example the editorial introduction in [AV] for an impression of the wide range of topics presently investigated.

More specifically, Popescu [Po1] developed a generalization of the Nagy-Foias dilation theory for row contractions and subsequently used this [Po2] to define a characteristic function for row contractions which is an operator on the free Fock space commuting with creation operators. Such operators are called *multi-analytic* and recently Popescu developed a noncommutative function theory for them, including an analogue of Nevanlinna-Pick interpolation [Po3]. An interesting further generalization involving Hilbert modules instead of Hilbert spaces and using operator algebraic methods is constructed by Muhly and Solel in [MS]. The natural question how to reconstruct a connection between this multivariate version and Lax-Phillips scattering theory has been taken up by Ball and Vinnikov in [BV].

We mention in this context the work of Bhat [Bh] and the book [Go1] of Gohm (principal investigator in this proposal) which already provides bridges to the theory of quantum open systems. In this project we want to go much further. One of the main aims of this project is to connect multivariate operator theory with a systematic theory of quantum control.

Kümmerer-Maassen Scattering Theory. To establish such a connection we want to make use of some work in quantum probability which at the present is not systematically integrated into the established schemes of quantum control. In [KM] Kümmerer and Maassen started a scattering theory for noncommutative Markov chains in the setting of von Neumann algebras with faithful normal states, motivated by Lax-Phillips scattering theory and by the fact that noncommutative Markov chains can be interpreted as automorphic dilations of their transition operators (completely positive maps), analogous to the Nagy-Foias dilation theory for contractions. A physical application of this theory is given in [WBKM] for the preparation of states in a micromaser interacting with a stream of atoms. Clearly this is a specific version of quantum open-loop control. Such a preparation is known to be always possible if the scattering theory is asymptotically complete.

The $\mathcal{B}(\mathcal{H})$-Version and Extensions. Gohm [Go1,Go2] showed that many aspects of Kümmerer-Maassen scattering theory can be simplified in the algebra $\mathcal{B}(\mathcal{H})$ of all bounded operators on a Hilbert space with a vector state. The general theory can be obtained from this case by GNS (Gelfand–Naimark-Segal) construction. An extension of the whole Markov chain can be realized on an infinite tensor product of Hilbert spaces. A key result says that asymptotic completeness is equivalent to the ergodicity of the extended transition operator. By this work the connections to operator theory have been strengthened and at the same time it has become easier to check asymptotic completeness computationally.

A Multi-Analytic Wave Operator. In recent work [DG] of Dey and Gohm there is the germ of a very specific and promising way to connect asymptotic completeness in the Kümmerer-Maassen scattering theory (in the version of [Go1]) with multivariate operator theory. Dilating a row contraction is equivalent to dilating a completely positive map as in [Bh], in this way it is related to noncommutative Markov chains. In [DG], Dey and Gohm compute a multi-analytic operator which corresponds to the wave operator in Kümmerer-Maassen scattering theory. In particular the correspondence between the free Fock space of Popescu's theory [Po2] and the infinite tensor product in [Go1] is made explicit. This is a generalization of Popescu's characteristic function, applicable to ergodic row contractions.

3 Programme and Methodology

The existing tools from operator theory have yet to be fully incorporated into the work on quantum control. Likewise the fascinating challenges of quantum control have been received limited interest by operator theorists. This is partly due to a 'cultural gap' between theoretical physicists who mainly drive the development of quantum control and the pure mathematicians responsible for the research in operator theory. The investigators of the current proposal have expertese and contacts with both communities and are therefore particularly qualified to bridge this gap.

These remarks are even more true if we consider recent developments in multivariate operator theory which seem to be more or less unknown to people working in quantum control. Function theoretic methods play a central role in classical control but there are difficulties to adapt them to the noncommutative mathematics needed for quantum control. We propose that some of these difficulties can be overcome by including the point of view of multivariate operator theory.

Hence the basic idea of this project is to"quantize" elements of classical control theory [Ki,Za] by replacing function theoretic methods with operator theoretic ones. This will be done with applications to quantum control. Multivariate operator theory and its noncommutative function theory offers the setting for this analysis. To get started we propose a concrete approach to these problems from a reformulation of the Kümmerer-Maassen scattering theory which seems particularly well suited to investigate this type of questions.

1. In a preparatory step, we will review existing results in Kümmerer-Maassen scattering theory in terms of their relevance to quantum control. The reasonable expectation is that this will naturally lead into the appropriate notion of quantum controllability.

2. We will work out more examples for asymptotically complete systems. Recently we started to examine connections with the work of Burgarth and Giovannetti on the up- and downloading of quantum information [BG] which suggests some new examples and new methods to check asymptotic completeness.

3. Using the framework of multi-analytic wave operators, we apply the noncommutative function theory developed by multivariate operator theory to the suitably reformulated problems of quantum control. Here we expect that suitably modified noncommutative methods will naturally extend the role of their classical counterparts to the new domain. Transferring these methods into quantum control will introduce new computational tools and create a new and more intuitive understanding of quantum control. In the converse direction, the modifications necessary to make operator theoretic tools applicable to the analysis of concrete quantum systems and experiments will indicate promising directions into the vast field of multivariate operator theory and will suggest new mathematics.

4. The quantum transfer function in [YK] and its generalizations in [GJ2,GGY] already provided some successful applications of function theory to quantum control in some specific (linear bosonic) models. We want to understand how this is related to the generalized characteristic function in [DG] which is a multi-analytic op-

erator. The infinite tensor products in [Go1,DG] can be thought of as a discretized version of the bosonic Fock space which gives a hint how this connection can be investigated. Both concepts generalize transfer functions/wave operators of classical control/scattering theory to a quantum setting. Understanding this connection is likely to be very helpful, in particular, to describing quantum concepts of controllability and observability.

5. Having established multi-analytic operators in quantum control we wish to make the existing work of Popescu [Po3] on noncommutative Nevanlinna-Pick interpolation available for quantum transfer techniques, and ultimately to build upon the operator theoretic tools which have already proved to be very important in H^∞-control in the classical case [Ki].

6. The tools of multivariate operator theory are much better developed for discrete time models and therefore it seems to be advisable to concentrate on discrete time systems in the beginning. Certainly the continuous time methods in [GJ1,GJ2,GGY] would profit from a discrete underpinning.

7. Once the bridge between quantum control and multivariate operator theory is understood in the specific directions described above we speculate that a considerable amount of related and deep mathematics becomes available for engineering applications.

The specific skills of the investigators and visiting researchers are listed below:

- Dr Gohm is an expert in operator theory with specific reference to Noncommutative Stochastic Processes and Ergodic Theory. He has developed a new field of non-commutative symbolic coding with Professor Kümmerer, and is currently working on applications to quantum memory with Daniel Burgarth.
- Professor Gough is an expert in Quantum Stochastic Processes and their applications to physical models and quantum control. He has developed the theory of quantum feedback networks in collaboration with Professor Matthew James.
- Professor Maassen is an expert in Quantum Probability, well-known for his pioneering work in the kernel calculus approach to describing quantum bosonic processes. He has more recently worked on filtering and purification of quantum systems.
- Professor Kümmerer has founded and developed the white noise approach to modelling quantum processes. Together with Hans Maassen, he has developed the "Kümmerer-Maassen".

3.1 Timeliness and Novelty

The is considerable interest in quantum control in the physics community and the subject has started to enter into conventional engineering journals and conferences. Recent theoretical investigations by Matthew James' group in Canberra has focussed on H^∞-control, and experimental feasibilty has been addressed by Hideo Mabuchi's team [Ma]. The development of a rigorous framework using the language of operator theory is both natural and timely, and we hope will open up new applications using this methodology.

4 Beneficiaries

The main beneficiaries will be the quantum control community. The research has a clear potential to influence design and implementation of control techniques for the specific purpose of quantum systems. This will be relevant to both theoreticians and experimentalists working in the area. As we have mentioned, robust control and H-infinity control techniques have influenced experimental design (for instance the work of Mabuchi and collaborators). The investigators seek to maintain their links with the theoreticians and experimentalists who will be the main end users of the research.

A successful programme will ensure that the Aberystwyth Quantum Control and Information research group establishes itself as a world-leading position in the area of applying advanced mathematical principles to the interdisciplinary field of quantum control. The investigators have strong reputations in field of mathematical modelling of open quantum systems, as well as a consistent record of successfully communicating these complex ideas to researchers in the physics and engineering world. They also have established research links with the potential users of the research. Their research should also generate new interest in the mathematics community, related to operator algebras. The group is intent on growing and is currently filling a funded PhD in the area of Quantum feedback control. The project will be of considerable benefit in setting up an active research group in a department that has recently undergone considerable up turn in staff numbers, and fits in with the Institute's research strategy. It will almost certainly impact on the materials physics academic and PDRA staff at the Institute in Aberystwyth University who are currently working in the nano-science area (quantum dots).

In the United Kingdom, there are several centres interested either in the quantum control/information or the mathematics of quantum stochastic processes, most notably at the universities of Nottingham, Cambridge and Lancaster. It is not unrealistic that the research output will also find interest with UK-based experimental physicists, particularly as this discipline is still relatively young and demonstrates the potential to become an important branch of technology, and the investigators would wish to bring their work to the attention of the wider scientific community and to help develop this emergent field.

5 Dissemination and Exploitation

The research wil be disseminated in internationally recognized journals: given the interdisciplinary nature of the research this will include Mathematics, Physics and Engineering journals as appropriate. The research will also we presented at dedicated Quantum Control conferences, such as the Principles and Applications of Control of Quantum Systems (PRACQSYS) workshop, to ensure that the results are disseminated directly to the users of the research.

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