GEOMETRIC PROPERTIES OF THE SET OF EXTENSIONS OF A STOCHASTIC MATRIX

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We consider stochastic matrices as restrictions of unital completely positive maps to diagonal subalgebras. The corresponding extensions of a stochastic matrix are classified by certain arrays of functionals and by positive definite kernels.

Introduction

The subject of extensions of completely positive maps given on operator algebras to the algebra of all bounded operators on a Hilbert space was first studied by Arveson [1]. For a survey see also the corresponding chapter in Effros/Ruan [4]. Recent work of Gohm [5,6] establishes some applications in quantum probability and indicates a way to more concrete descriptions by a duality with dilation theory.

Many interesting questions arise at this point, and in this paper we provide some playground by explicitly describing the most elementary case, namely that of stochastic matrices. This is also of interest for its own sake because embeddings of classical Markov chains into noncommutative ones are a natural topic for quantum probabilists.

Surprisingly it turns out that a sort of geometric reasoning is appropiate here which has a natural starting point in Arveson's notion of a metric operator space [2]. We review it in Section 1 and give an alternative formulation by representing functionals. In Section 2 we define realizations of stochastic matrices and show that realizations are representing functionals which classify the extensions to normal unital completely positive maps on all bounded operators. The geometric character becomes explicit by a bijective correspondence with certain positive definite kernels.

In Section 3 we include measures and states into the extension procedure. This is the setting of Gohm [5,6] and we show how the duality theory between extensions and dilations can be described very concretely in terms of realizations. Again there is a geometric picture given by certain positive definite kernels. A remarkable conclusion tells us that the extension set essentially only depends on the pattern of zeroes of the stochastic matrix. Details on that can be found in the remarks at the end of the paper.

The following notation will be used invariably throughout the paper. $\mathcal{P}, \mathcal{G}, \mathcal{H}$ denote (complex, separable) Hilbert spaces with ONBs $\{\epsilon_k\}_{k=1}^d, \{\delta_j^{\mathcal{G}}\}_{j=1}^n, \{\delta_i^{\mathcal{H}}\}_{i=1}^m$ (the superscripts will be omitted from now on). d, n, m may be finite or ∞. Operators are identified with matrices relative to these ONBs. Inner products are linear in the second variable.

1. Metric operator spaces and their duals

We start with the objects given at the end of the introduction. Now assume that $\{a_k\}_{k=1}^d \subset \mathcal{B}(\mathcal{G}, \mathcal{H})$ (bounded linear maps from $\mathcal{G}$ to $\mathcal{H}$) satisfy $\sum_{k=1}^d \|a_k^* \xi\|^2 < \infty$ for all $\xi \in \mathcal{H}$ and that the a_k are linearly independent. (For $d = \infty$ the sum $\sum_{k=1}^d \lambda_k a_k$ for $(\lambda_k) \in l^2$ is always strongly convergent and linear independence means that in this situation $\sum_{k=1}^d \lambda_k a_k = 0$ implies $\lambda_k = 0$ for all k. See Arveson [2], 9.1.) Then $\epsilon_k \mapsto a_k$ for $k = 1, \ldots, d$ extends to an injective linear map $a : \mathcal{P} \to \mathcal{B}(\mathcal{G}, \mathcal{H})$. The map a can be used to transfer the inner product to $a(\mathcal{P}) \subset \mathcal{B}(\mathcal{G}, \mathcal{H})$, which in this way becomes a metric operator space in the sense of Arveson. In fact, Arveson [2] considers $\mathcal{G} = \mathcal{H}$ but his arguments apply also here.

In particular, from a metric operator space we can define a normal completely positive map $Z : \mathcal{B}(\mathcal{G}) \to \mathcal{B}(\mathcal{H})$ by $Z(x) = \sum_{k=1}^d a_k x a_k^*$, where the r.h.s. is called a minimal Kraus decomposition for Z. All other minimal Kraus decompositions are unitarily equivalent, i.e., if also $Z(x) = \sum_{k=1}^d b_k x b_k^*$ with $\{b_k\}_{k=1}^d \subset \mathcal{B}(\mathcal{G}, \mathcal{H})$ and the b_k are linearly independent then $b = a\,u$, where $b : \epsilon_k \mapsto b_k$ for all k and $u \in \mathcal{B}(\mathcal{P})$ is unitary. This yields a bijective correspondence between normal completely positive maps $Z : \mathcal{B}(\mathcal{G}) \to \mathcal{B}(\mathcal{H})$ and metric operator spaces in $\mathcal{B}(\mathcal{G}, \mathcal{H})$. Unital maps are characterized by $\sum_{k=1}^d \|a_k^* \xi\|^2 = \|\xi\|^2$ for all ξ or equivalently $\sum_{k=1}^d a_k a_k^* = \mathbb{1}$ (weakly or strongly).

Let $\mathcal{P}^\star$ be the dual of $\mathcal{P}$. Avoiding immediate identification with $\mathcal{P}$ will give us some conceptual advantages. $\mathcal{P}^\star$ contains in particular the linear functionals

$$a^{ij} : \mathcal{P} \to \mathbb{C}, \quad \xi \mapsto \langle \delta_i, a(\xi) \delta_j \rangle.$$

When the inner product of $\mathcal{P}$ is transferred from $\mathcal{P}$ to $\mathcal{P}^\star$ the dual basis

$\{\epsilon^k\}_{k=1}^d$ of $\{\epsilon_k\}_{k=1}^d$ becomes an ONB of $\mathcal{P}^\star$. With $a_k^{ij} := a^{ij}(\epsilon_k) = \langle\delta_i, a_k\delta_j\rangle$ we get the formula

$$a^{ij} = \sum_{k=1}^{d} a_k^{ij}\epsilon^k.$$

For $d = \infty$, note that $\sum_{k=1}^d \|a_k^*\xi\|^2 < \infty$ implies that $(a_k^{ij})_{k=1}^d \in l^2$, i.e., the sum above is convergent. The inner product $\langle a^{ij}, a^{i'j'}\rangle$ in $\mathcal{P}^\star$ is nothing but the standard inner product of the vectors $(a_k^{ij})_{k=1}^d$ and $(a_k^{i'j'})_{k=1}^d$ in $\mathbb{C}^d$ or l^2.

The linear independence of the a_k implies that the a^{ij} ($i = 1,\ldots,m$ and $j = 1,\ldots,n$) are total in $\mathcal{P}^\star$. In fact, if $b = \sum_{k=1}^d \beta_k\epsilon^k$ is orthogonal to all a^{ij} then

$$\langle\delta_i, \sum_{k=1}^{d}\overline{\beta}_k a_k\delta_j\rangle = \sum_{k=1}^{d}\overline{\beta}_k\langle\delta_i, a_k\delta_j\rangle = \sum_{k=1}^{d}\overline{\beta}_k a_k^{ij} = \langle b, a^{ij}\rangle = 0$$

i.e., $\sum_{k=1}^d \overline{\beta}_k a_k = 0$ and thus $\beta_k = 0$ for all k and finally $b = 0$.

The argument can be reversed. For $d < \infty$ we can start with a Hilbert space $\mathcal{P}$ and any total array $a^{ij} \in \mathcal{P}^\star$ ($i = 1,\ldots,m$ and $j = 1,\ldots,n$) and construct backwards a metric operator space from that. For $d = \infty$ the property $\sum_{k=1}^d \|a_k^*\xi\|^2 < \infty$ is an additional assumption. This motivates the following

Definition 1.1. An array $(a^{ij})_{i,j}$ of vectors in a Hilbert space $\mathcal{P}^\star$ is said to *represent* a metric operator space or the corresponding map if it is total in $\mathcal{P}^\star$ and if for an ONB $\{\epsilon_k\}$ in the dual $\mathcal{P}$ the formula $\langle\delta_i, a_k\delta_j\rangle := a^{ij}(\epsilon_k)$ defines operators $\{a_k\}_{k=1}^d \subset \mathcal{B}(\mathcal{G},\mathcal{H})$ such that $\sum_{k=1}^d \|a_k^*\xi\|^2 < \infty$ for all $\xi \in \mathcal{H}$.

One can check that this does not depend on the choice of the ONB $\{\epsilon_k\}$ in $\mathcal{P}$. A representing array always represents a normal completely positive map $Z : \mathcal{B}(\mathcal{G}) \to \mathcal{B}(\mathcal{H})$ via the corresponding minimal Kraus representation.

From the uniqueness result for the a_k quoted above we conclude immediately that two arrays (a^{ij}) and (b^{ij}) represent the same normal completely positive map $Z : \mathcal{B}(\mathcal{G}) \to \mathcal{B}(\mathcal{H})$ if and only if there is a unitary $u \in \mathcal{B}(\mathcal{P}^\star)$ such that $b^{ij} = u\,a^{ij}$ for all i, j. If we say that a geometric property is something that does not change under unitary transformations then an interesting way to think about normal completely positive maps geometrically consists in looking at a representing array (a^{ij}) in $\mathcal{P}^\star$. This tool has been used ad hoc and in special cases ($n = m = 2$) by Kümmerer [8] and by Gohm [6] and will be used more systematically in the following.

2. Realizations of stochastic matrices

By a stochastic $m \times n-$matrix we mean a matrix with nonnegative entries with all row sums equal to one (convergent if $n = \infty$). The following definition prepares the study of extensions of stochastic matrices.

Definition 2.1. If an array (a^{ij}) $(i = 1, \ldots, m$ and $j = 1, \ldots, n)$ of vectors in a Hilbert space $\mathcal{P}^\star$ satisfies

(a) $T_{ij} = \|a^{ij}\|^2$
(b) $a^{ij} \perp a^{i'j}$ if $i \neq i'$

for all i, j, then we say that (a^{ij}) is a *realization* of the $m \times n-$matrix $T = (T_{ij})$.

Realizations always exist. For example, we may choose all vectors a^{ij} orthogonal to each other with length according to (a). Note that if u is unitary or antiunitary on $\mathcal{P}^\star$, then (a^{ij}) realizes the same matrix as $(u\, a^{ij})$. To classify realizations of a matrix T geometrically we can proceed as follows. Given a matrix T with nonnegative entries let Q_j (with $j = 1, \ldots, n$) be a Hilbert space spanned by vectors ξ_{ij} $(i = 1, \ldots, m)$ satisfying $T_{ij} = \|\xi_{ij}\|^2$ and $\xi_{ij} \perp \xi_{i'j}$ if $i \neq i'$. Note that the dimension of Q_j equals the number of nonzero entries in the $j-$th column of T and that also the vectors ξ_{ij} are determined by T up to unitary equivalence.

To any realization (a^{ij}) of T we associate a positive definite kernel K with values $K(r, s) \in \mathcal{B}(Q_s, Q_r)$, i.e.,

$$\sum_{r,s=1}^{n} \langle \xi_r, K(r, s)\xi_s \rangle \geq 0$$

for all (ξ_l) with finite support. See for example Constantinescu [3] for details on positive definite kernels. The properties of the ξ_{ij} are chosen in such a way that

$$v_j : Q_j \to \mathcal{P}^\star, \quad \xi_{ij} \mapsto a^{ij}$$

extends to an isometry and we define $K(r, s) := v_r^* \, v_s$. Then K is positive definite and satisfies $K(r, r) = \mathbb{1}$ for all r. If $u \in \mathcal{B}(\mathcal{P}^\star)$ is unitary then $(u\, a^{ij})$ has the same associated kernel, i.e., we have indeed constructed a geometric property. We call K the *correlation kernel* of the realization (a^{ij}).

Theorem 2.1. *Let T be a stochastic $m \times n-$matrix. Every realization of T represents a normal unital completely positive map. There is a bijective correspondence between*

(1) unitary equivalence classes of realizations of T
(2) normal unital completely positive maps $Z : \mathcal{B}(\mathcal{G}) \to \mathcal{B}(\mathcal{H})$ which map the diagonal into the diagonal such that the restricted map is given by T
(3) positive definite kernels K with $K(r,s) \in \mathcal{B}(Q_s, Q_r)$ and $K(r,r) = \mathbb{I}$ for $r, s \in \{1, \ldots, n\}$.

If (a^{ij}) is a realization of T, then Z is the map represented by it and K is the associated correlation kernel.

Proof: Suppose that (a^{ij}) is a realization of T in $\mathcal{P}^\star$. Choose an ONB $\{\epsilon_k\}_{k=1}^d$ of $\mathcal{P}$ and define $a_k^{ij} := a^{ij}(\epsilon_k)$. On $\xi = \sum_{i=1}^m \lambda_i \delta_i \in \mathcal{H}$ (with $(\lambda_i) \in l^2$ if $m = \infty$) we can define bounded operators

$$a_k^* : \mathcal{H} \to \mathcal{G}, \quad \xi \mapsto \sum_{i=1}^m \sum_{j=1}^n \lambda_i a_k^{ij} \delta_j$$

(in particular $\langle \delta_i, a_k \delta_j \rangle = a_k^{ij}$). In fact, we have

$$\sum_k ||a_k^* \xi||^2 = \sum_k || \sum_{i,j} \lambda_i a_k^{ij} \delta_j ||^2 = \sum_{k,j} | \sum_i \lambda_i a_k^{ij} |^2 = \sum_{j,k} |(\sum_i \lambda_i a^{ij})(\epsilon_k)|^2$$

$$= \sum_j || \sum_i \lambda_i a^{ij} ||^2 = \sum_{i,j} |\lambda_i|^2 ||a^{ij}||^2 = \sum_i |\lambda_i|^2 = ||\xi||^2,$$

i.e., $\sum_k ||a_k^* \xi||^2 = ||\xi||^2$ for all ξ, which characterizes a metric operator space corresponding to a normal unital completely positive map, see Section 1.

Let $Z : \mathcal{B}(\mathcal{G}) \to \mathcal{B}(\mathcal{H})$ be a normal completely positive map, given by $Z(x) = \sum_{k=1}^d a_k x a_k^*$ with $\{a_k\}_{k=1}^d \subset \mathcal{B}(\mathcal{G}, \mathcal{H})$. We compute its action on a diagonal matrix x with diagonal entries $x_1, \ldots, x_n$:

$$\Big(\sum_{k=1}^d a_k \, x \, a_k^*\Big)_{ii'} = \sum_{k,j} (a_k)_{ij} \, x_j \, (a_k^*)_{ji'} = \sum_j \langle a^{i'j}, a^{ij} \rangle \, x_j .$$

Here we used the description of Z by the representing array (a^{ij}) in $\mathcal{P}^\star$. Thus $\langle a^{i'j}, a^{ij} \rangle = 0$ for $i \neq i'$ is equivalent to the fact that Z maps diagonal matrices into diagonal matrices. In this case the restricted map is given by a matrix T with entries $T_{ij} = \langle a^{ij}, a^{ij} \rangle = ||a^{ij}||^2$. That different arrays represent the same map Z iff they are unitarily equivalent has been noted in the previous section. We conclude that (1) and (2) are related by a bijection.

Given (1), it has already been shown above how to associate the correlation kernel K in (3) which only depends on the unitary equivalence class. Conversely, given a positive definite kernel K as in (3), there is a minimal Kolmogorov decomposition [3] for it, i.e., there exists a Hilbert space Q and $v_j \in \mathcal{B}(Q_j, Q)$ (for $j = 1, \ldots, n$) such that $K(r,s) := v_r^* v_s$. Because $K(r,r) = \mathbb{1}$ for all r the v_j are isometries. Then putting $a^{ij} := v_j(\xi_{ij})$ for all i, j we can identify Q with $\mathcal{P}^\star$ and obtain a realization (a^{ij}) of T with correlation kernel K. A minimal Kolmogorov decomposition is unique up to unitary equivalence which implies that two arrays (a^{ij}) and (b^{ij}) obtained from the same kernel are unitarily equivalent. Thus we have also a bijection between (1) and (3). □

Theorem 2.1 characterizes a certain set in three different ways. It shows that realizations of stochastic matrices classify certain extensions of them and that this can be mapped bijectively onto a well known set of kernels. We mention two structures on this set which are now immediate. First, with the natural affine structure from (2) or (3), it is a convex set. Second, observe that antiunitary transformations on $\mathcal{P}^\star$ in general change the unitary equivalence class in (1). Explicitly, if $a^{ij} = \sum_{k=1}^d a_k^{ij} \epsilon^k$ for an ONB $\{\epsilon^k\}$ then $\overline{a}^{ij} = \sum_{k=1}^d \overline{a}_k^{ij} \epsilon^k$ (complex conjugation) may be called a conjugate realization of T. Because two antiunitaries differ by a unitary, the unitary equivalence class obtained in this way does not depend on the choice of the ONB $\{\epsilon^k\}$ and thus we have a canonical antilinear inversion on our convex set which we call *conjugation*. The conjugate extension $\overline{Z}$ is given by $\overline{Z}(x) = \overline{Z(\overline{x})}$. where on the r.h.s. we have complex conjugation of all matrix entries. Similarly we also have a conjugate kernel $\overline{K}$.

3. Extension of states included

For probabilistic applications it is essential to include states into the extension procedure of the previous section, see Gohm [6]. Assume now that T is a stochastic $m \times n$−matrix and let $\nu = (\nu_j)_{j=1}^n$ and $\mu = (\mu_i)_{i=1}^m$ be probability measures on $\{1, \ldots, n\}$ and $\{1, \ldots, m\}$ such that all ν_j and μ_i are nonzero and we have $\mu \circ T = \nu$. Such probability measures can be found iff T has at least one nonzero entry in each column. So this is an implicit assumption on T from now on. We have a *dual* stochastic $n \times m$−matrix

$T' = (T'_{ji})$ given by the equation

$$\mu_i \, T_{ij} \;=\; \nu_j \, T'_{ji} \quad \text{for all } i, j.$$

Then $\nu \circ T' = \mu$ and $(T')' = T$.

We can think of the Hilbert spaces $\mathcal{G}$ and $\mathcal{H}$ as the GNS- Hilbert spaces for the (commutative von Neumann) algebras $\mathcal{A}$ and $\mathcal{B}$ of bounded functions on $\{1, \ldots, n\}$ and $\{1, \ldots, m\}$ for the states induced by ν and μ. Note that $T : \mathcal{A} \to \mathcal{B}$ and $T' : \mathcal{B} \to \mathcal{A}$. Let us write $\Omega_\nu \in \mathcal{G}$ and $\Omega_\mu \in \mathcal{H}$ for the corresponding cyclic vectors. Explicitly, we have $\Omega_\nu = \sum_{j=1}^n \sqrt{\nu_j} \, \delta_j$ and $\Omega_\mu = \sum_{i=1}^m \sqrt{\mu_i} \, \delta_i$, where the basis vectors δ_i, δ_j in $\mathcal{H}, \mathcal{G}$ are realized by characteristic functions of single points which are also elements in $\mathcal{B}, \mathcal{A}$ and denoted as such in the same way. This should not cause confusion.

Now let (a^{ij}) $(i = 1, \ldots, m$ and $j = 1, \ldots, n)$ be a realization of T and let $Z : \mathcal{B}(\mathcal{G}) \to \mathcal{B}(\mathcal{H})$ with $Z(x) = \sum_{k=1}^d a_k x a_k^*$ be the corresponding completely positive extension, according to Theorem 2.1.

Lemma 3.1. *The following assertions are equivalent:*

(1) $\langle \Omega_\nu, x \, \Omega_\nu \rangle \;=\; \langle \Omega_\mu, Z(x) \, \Omega_\mu \rangle$ *for all* $x \in \mathcal{B}(\mathcal{G})$
(2) $a_k^* \, \Omega_\mu = \overline{\omega}_k \, \Omega_\nu$ *for some* $\omega_1, \ldots, \omega_d \in \mathbb{C}$.
(3) There exists a unit vector $\Omega^* = \sum_{k=1}^d \omega_k \, \epsilon^k \in \mathcal{P}^\star$ *satisfying*

$$\Omega^* = \sum_{i=1}^n \sqrt{\frac{\mu_i}{\nu_j}} \, a^{ij} \quad (\textit{for all}\, j).$$

Proof: These elementary equivalences are shown in a more general context in Gohm [6], A.5. To relate it to our setting we prove the equality

$$\sum_{k=1}^d \omega_k \, \epsilon^k = \sum_{i=1}^n \sqrt{\frac{\mu_i}{\nu_j}} \, a^{ij}$$

for all j, where the ω_k satisfy (2). In fact, from

$$\overline{\omega}_k \sum_{j=1}^n \sqrt{\nu_j} \, \delta_j = \overline{\omega}_k \Omega_\nu = a_k^* \, \Omega_\mu = \sum_{j,i} (a_k^*)_{ji} \sqrt{\mu_i} \, \delta_j$$

we obtain

$$\overline{\omega}_k \sqrt{\nu_j} = \sum_i (a_k^*)_{ji} \sqrt{\mu_i}$$

and hence

$$\sqrt{\nu_j} \sum_k \omega_k \, \epsilon^k = \sum_{i,k} a_k^{ij} \sqrt{\mu_i} \, \epsilon^k = \sum_i \sqrt{\mu_i} \, a^{ij}. \quad \square$$

For the following considerations it is convenient to introduce vectors $a_{ij} \in \mathcal{P}$ by

$$\langle a_{ij}, \xi \rangle = a^{ij}(\xi) \quad \text{for all } \xi \in \mathcal{P},$$

i.e., $a_{ij} = \sum_{k=1}^{d} \overline{a}_k^{ij} \epsilon_k$. Let us say that (a^{ij}) is a *$\mu-$realization* of T if the conditions in Lemma 3.1 are satisfied. From now on we assume this. Then we also have a unit vector $\Omega \in \mathcal{P}$ given by $\Omega := \sum_{i=1}^{n} \sqrt{\frac{\mu_i}{\nu_j}}\, a_{ij}$.

Proposition 3.1. *Let (a^{ij}) be a $\mu-$realization of T. Then the map*

$$\begin{aligned} J : \mathcal{B} &\to \mathcal{A} \otimes \mathcal{B}(\mathcal{P}), \\ \delta_i &\mapsto \sum_{j=1}^{n} \delta_j \otimes \frac{|a_{ij}\rangle\langle a_{ij}|}{\|a_{ij}\|^2} \quad (i = 1, \ldots, m) \end{aligned}$$

is a normal $^*-$homomorphism such that

$$\big(Id_{\mathcal{A}} \otimes \langle \Omega, \cdot\, \Omega\rangle\big) \circ J \;=\; T'.$$

We have used von Neumann tensor products and Dirac notation for rank one operators. In the language of Gohm [5,6], Proposition 3.1 tells us that J is a *weak tensor dilation* of T'.

Proof: Because $a^{ij} \perp a^{i'j}$ for $i \neq i'$, we find that $\frac{|a_{ij}\rangle\langle a_{ij}|}{\|a_{ij}\|^2}$ are one-dimensional projections which are orthogonal for different i. Thus, writing $J(b) = \sum_{j=1}^{n} \delta_j \otimes J_j(b)$, we see that the J_j are representations of $\mathcal{B}$ on $\mathcal{P}$. We conclude that J is a normal $^*-$homomorphism. Further we get

$$\begin{aligned} \big(Id_{\mathcal{A}} \otimes \langle \Omega, \cdot\, \Omega\rangle\big) \circ J(\delta_i) \;&=\; \sum_{j=1}^{n} \delta_j \, \frac{|\langle \Omega, a_{ij}\rangle|^2}{\|a_{ij}\|^2} \\ = \sum_{j=1}^{n} \delta_j \, \frac{\mu_i}{\nu_j} \|a_{ij}\|^2 \;&=\; \sum_{j=1}^{n} \delta_j \, \frac{\mu_i}{\nu_j} T_{ij} \;=\; \sum_{j=1}^{n} \delta_j \, T'_{ji}, \end{aligned}$$

i.e., $\big(Id_{\mathcal{A}} \otimes \langle \Omega, \cdot\, \Omega\rangle\big) \circ J \;=\; T'$. □

Proposition 3.1 is a concrete instance of a much more general phenomenon revealed in Gohm [5,6], compare also Gohm/Skeide [7]. Namely, whenever T is a normal unital completely positive map between von Neumann algebras $\mathcal{A}$ and $\mathcal{B}$, there is a correspondence between extensions of T respecting some previously fixed states and weak tensor dilations of the dual map T'. In our setting, where T is a stochastic matrix, starting from the weak tensor dilation J above we can construct an *associated isometry*

$$v : \mathcal{H} \to \mathcal{G} \otimes \mathcal{P}, \quad b\,\Omega_\mu \mapsto J(b)(\Omega_\nu \otimes \Omega) \quad (b \in \mathcal{B}),$$

explicitly: $\sqrt{\mu_i}\, v(\delta_i) = v(\delta_i \Omega_\mu) = \sum_{j=1}^n \sqrt{\nu_j}\, \delta_j \otimes \sqrt{\frac{\mu_i}{\nu_j}}\, a_{ij}$, i.e., we get the simple formula

$$v : \delta_i \mapsto \sum_{j=1}^n \delta_j \otimes a_{ij}.$$

With $a_{ij} = \sum_{k=1}^d \overline{a}_k^{ij} \epsilon_k$ this can be written as

$$v\, \delta_i = \sum_{j,k} \delta_j \otimes \overline{a}_k^{ij} \epsilon_k = \sum_{j,k} \overline{a}_k^{ij} \delta_j \otimes \epsilon_k = \sum_{k=1}^d a_k^*(\delta_i) \otimes \epsilon_k,$$

i.e. $v\,\xi = \sum_{k=1}^d a_k^*(\xi) \otimes \epsilon_k$ for all $\xi \in \mathcal{H}$. Thus the corresponding extension Z of T can be written as

$$Z(x) = v^*\; x \otimes \mathbb{1}\; v = \sum_{k=1}^d a_k\, x\, a_k^*,$$

and we have reconstructed the extension from the dilation.

Our emphasis here is on the fact that both extensions and dilations of stochastic matrices are very conveniently described in terms of realizations. With a suitable equivalence relation on weak tensor dilations discussed in Gohm [6], 1.4, this correspondence between extensions of T (respecting fixed states) and (equivalence classes of) weak tensor dilations of T' becomes a bijection. It was noted in Gohm [5,6] that the set $\mathcal{Z}$ characterized in this way is convex and closed in suitable topologies.

In our setting, with T a stochastic matrix, we can combine Theorem 2.1 and Lemma 3.1 to see that the set $\mathcal{Z} = \mathcal{Z}(T, \mu)$ can be described geometrically as the set of unitary equivalence classes of $\mu-$realizations of T, i.e., realizations (a^{ij}) of T in $\mathcal{P}^\star$ for which $\sum_{i=1}^n \sqrt{\frac{\mu_i}{\nu_j}}\, a^{ij} (= \Omega^*)$ does not depend on j. We notice that on $\mathcal{Z}$ there is a canonical conjugation, inherited from the one discussed in Section 2. A description of $\mathcal{Z}$ by kernels can be obtained as follows.

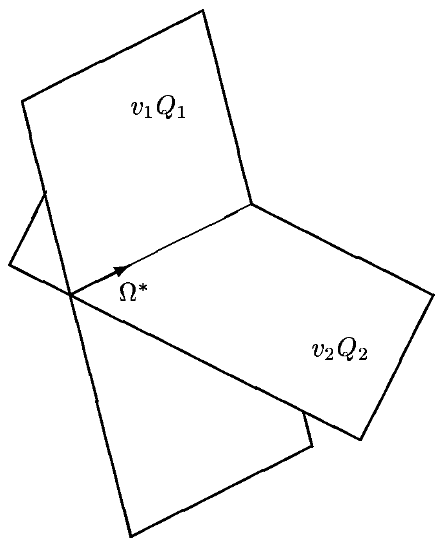
We start with the Hilbert spaces Q_j $(j = 1, \ldots, n)$ defined in Section 2. They are spanned by vectors ξ_{ij} $(i = 1, \ldots, m)$ satisfying $T_{ij} = \|\xi_{ij}\|^2$ and $\xi_{ij} \perp \xi_{i'j}$ if $i \neq i'$. Since T satisfies $\mu \circ T = \nu$, there are unit vectors $\Omega_j \in Q_j$ given by $\Omega_j := \sum_{i=1}^n \sqrt{\frac{\mu_i}{\nu_j}}\, \xi_{ij}$. Because now (a^{ij}) is a $\mu-$realization, the isometries $v_j : Q_j \to \mathcal{P}^\star$, $\xi_{ij} \mapsto a^{ij}$ map the Ω_j to $\Omega^* \in \mathcal{P}^\star$. We define $Q_j^0 := Q_j \ominus \mathbb{C}\, \Omega_j$ and $v_j^0 := v_j|_{Q_j^0}$ $(j = 1, \ldots, n)$ and for all $r, s \in \{1, \ldots, n\}$

$$K^0(r,s) := (v_r^0)^* v_s^0 \in \mathcal{B}(Q_s^0, Q_r^0).$$

Then K^0 is a positive definite kernel with $K^0(r,r) = \mathbb{1}$ for all r which we call the *restricted correlation kernel.* If we take another $\mu-$realization (b^{ij}) then because $b^{ij} = u\, a^{ij}$ for a unitary $u \in \mathcal{B}(\mathcal{P}^\star)$ the restricted correlation kernel is not changed and it represents therefore a geometric property.

Proposition 3.2. *There is a bijective correspondence between the set $\mathcal{Z} = \mathcal{Z}(T,\mu)$ and the set of positive definite kernels K^0 with $K^0(r,s) \in \mathcal{B}(Q^0_s, Q^0_r)$ and $K^0(r,r) = \mathbb{1}$ for all $r,s \in \{1,\ldots,n\}$. It is given by the restricted correlation kernel.*

Proof: We only have to modify the proof of Theorem 2.1 in a suitable way. We have already seen how to produce a restricted correlation kernel K^0 from a $\mu-$realization corresponding to $Z \in \mathcal{Z}$. Conversely, given a kernel K^0, we use it to agglutinate the spaces Q_j in such a way that all the Ω_j become a single vector Ω^*.



In detail, using a minimal Kolmogorov decomposition for the kernel K^0, we get a Hilbert space Q^0 and isometries $v^0_j : \mathcal{B}(Q^0_j, Q^0)$ such that $K^0(r,s) = (v^0_r)^* v^0_s$ for all $r,s \in \{1,\ldots,n\}$. We extend them to isometries $v_j \in \mathcal{B}(Q_j, Q^0 \oplus \mathbb{C}\Omega^*)$ by $v_j \Omega_j := \Omega^*$ for all j, where Ω^* is a unit vector. Then it is easily checked that $a^{ij} := v_j\, \xi_{ij}$ for all i,j and a corresponding identification of $Q^0 \oplus \mathbb{C}\Omega^*$ with $\mathcal{P}^\star$ defines a $\mu-$realization of T, and K^0 is the corresponding restricted correlation kernel. By the uniqueness property of minimal Kolmogorov decompositions two $\mu-$realizations arising from the same kernel are unitarily equivalent and belong to the same $Z \in \mathcal{Z}$. Hence $Z \mapsto K^0$ is a bijection. $\square$

Let us finish with some remarks which should help to put these results into perspective.

To determine the kernels in Proposition 3.2 the only relevant information about the spaces Q_j^0 is their dimension which equals the number of nonzero entries in the $j-$th column of T minus one. Of course, in the case where this dimension is zero the corresponding entries of K^0 also have to be zero.

There is always a distinguished solution given by a kernel K^0 with $K^0(r,s) = 0$ for $r \neq s$. When in T there are less than two columns with two or more nonzero entries then this is the only solution. The fact that $\mathcal{Z}$ is always nonempty has been observed in a more general context in Gohm/Skeide [7].

The correspondence established in Proposition 3.2 yields the surprising fact that the set $\mathcal{Z}$ depends only rather slightly on the details of T and T'. More precisely, though obviously the extensions $Z : \mathcal{B}(\mathcal{G}) \to \mathcal{B}(\mathcal{H})$ depend on all entries T_{ij} and on the measures μ and ν (or on T and T'), we have shown that whenever for two stochastic $m \times n-$matrices T_1 and T_2 the number of nonzero entries in the $j-$th column coincides for each j then there is a canonical bijection between the extension sets $\mathcal{Z}_1$ and $\mathcal{Z}_2$, via the correspondence to kernels given in Proposition 3.2.

Interpreting $\mathcal{Z}$ as the set of (equivalence classes of) weak tensor dilations of T', see Proposition 3.1, we can give the following interpretation of the restricted correlation kernel. Using the notation introduced in Proposition 3.1 and its proof, we can consider the relative position of the commutative algebras $J_j(\mathcal{B})$ in $\mathcal{B}(\mathcal{P})$ $(j = 1, \dots, n)$ by looking at the corresponding subspaces $J_j(\mathcal{B})\Omega \subset \mathcal{P}$. The formula $v : \delta_i \mapsto \sum_{j=1}^{n} \delta_j \otimes a_{ij}$ for the associated isometry makes clear that it is this relative position that is described by the (restricted) correlation kernel.

It is very instructive to check all these observations in the simple example of $2 \times 2-$stochastic matrices with nonzero entries. Then $n = m = 2$

and the kernels K^0 are $\mathbb{C}$−valued and parametrized by the off-diagonal entry $c := K^0(2,1)$, i.e., by a complex parameter c with $|c| \leq 1$. This parametrization of $\mathcal{Z}$ has already been observed in Gohm [6], where also the corresponding extensions $Z : M_2 \to M_2$ and dilations $J : \mathbb{C}^2 \to \mathbb{C}^2 \otimes \mathcal{B}(\mathcal{P})$ are explicitly computed.

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