Contemporary Mathematics

A Duality between Extension and Dilation

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ABSTRACT. For normal unital completely positive maps on von Neumann algebras respecting distinguished states, we consider the problem to find normal unital completely positive extensions acting on all bounded operators on the GNS-spaces and respecting the corresponding cyclic vectors. We show that there exists a duality relating this problem to a certain dilation problem on the commutants. Some explicit computations for low dimensions are presented.

Introduction

The following considerations concern normal unital completely positive maps on von Neumann algebras. The algebras are represented on Hilbert spaces where the relevant states are represented by vectors. We refer to [**Sa**] and [**Ta**] for terminology and standard results about these objects.

While much is known about extensions of such maps on the one hand and dilations of them on the other hand, we develop a specific connection between these fields of study. We formulate our extension problem in section 2 and our dilation problem in section 3. Both involve in addition to the maps certain distinguished states. Our main result is then presented in section 4, and it tells us that dilations/extensions of the map on the algebra correspond to extensions/dilations of the dual map on the commutant. For standard representations we have a bijective correspondence if we introduce a suitable equivalence relation on the dilations. The final form is given in Theorem 4.4. We close by reconsidering the elementary example of section 1 to illustrate the theory.

We think that these results have some intrinsic interest for the theory of completely positive maps and this is the point of view from which they are developed here. But let us remark that the starting point of our investigations which led eventually to the condensed version presented here are some questions in non-commutative probability. Due to lack of space we cannot show these applications here and only give the following hints: In non-commutative probability normal unital completely positive maps represent transition operators of non-commutative Markov processes. The dilation problem is related to the construction of the Markov processes and to their structural properties, see [**BP, KM**], while the extension

1991 *Mathematics Subject Classification.* Primary 47B65, 47L30; Secondary 60J10.
Key words and phrases. completely positive, extension, dilation.
The author was supported in part by the DFG.

problem is related to extension of Markov processes to larger ones. The duality shown in this paper has proved to be useful in particular for the Kümmerer-Maassen scattering theory of Markov processes, see [**KM**]. More details can be found in [**Go**].

1. An Extension Problem (Example)

Here we present the extension problem to be considered in the most elementary nontrivial case. Consider a stochastic matrix $\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. We think of it as an operator

$$S : \mathbb{C}^2 \to \mathbb{C}^2, \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \frac{1}{2}\begin{pmatrix} x_1 + x_2 \\ x_1 + x_2 \end{pmatrix}.$$

There is an invariant probability measure $\left(\frac{1}{2}, \frac{1}{2}\right)$, giving rise to an invariant state ϕ in the sense that $\phi \circ S = \phi$.

We can apply the GNS-construction for the algebra $\mathcal{A} = \mathbb{C}^2$ with respect to the state ϕ, and we get the Hilbert space $\mathcal{H} = \mathbb{C}^2$ (with canonical scalar product) and the unit vector $\Omega = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Now the state ϕ is realized as a vector state, i.e. $\phi\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = <\Omega, \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix}\Omega>$. Identifying $\mathcal{A} = \mathbb{C}^2$ with the diagonal subalgebra of $\mathcal{B}(\mathcal{H}) = M_2$ we have

$$S : \quad x = \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix} \mapsto \frac{1}{2}\begin{pmatrix} x_1 + x_2 & 0 \\ 0 & x_1 + x_2 \end{pmatrix}$$

and $<\Omega, x\Omega> = <\Omega, S(x)\Omega>$. We shall now ask for completely positive maps $Z : M_2 \to M_2$ which extend S and satisfy $<\Omega, x\Omega> = <\Omega, Z(x)\Omega>$ for all $x \in M_2$.

Let us try a direct approach. Any completely positive map $Z : M_2 \to M_2$ can be written in the form $Z(x) = \sum_{k=1}^d a_k\, x\, a_k^*$ with $a_k \in M_2$. Introduce four vectors $a_{ij} \in \mathbb{C}^d,\ i,j = 1,2$, whose k-th entry is the ij-entry of a_k. With the canonical scalar product and euclidean norm on $\mathbb{C}^d$ we get by direct computation:

$$Z\begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix} = \begin{pmatrix} \|a_{11}\|^2 x_1 + \|a_{12}\|^2 x_2 & <a_{21}, a_{11}> x_1 + <a_{22}, a_{12}> x_2 \\ <a_{11}, a_{21}> x_1 + <a_{12}, a_{22}> x_2 & \|a_{21}\|^2 x_1 + \|a_{22}\|^2 x_2 \end{pmatrix}$$

If Z is an extension of S we conclude that

$$\|a_{11}\|^2 = \|a_{12}\|^2 = \|a_{21}\|^2 = \|a_{22}\|^2 = \frac{1}{2},$$
$$< a_{11}, a_{21} > = < a_{12}, a_{22} > = \ 0.$$

Now $<\Omega, x\Omega> = <\Omega, Z(x)\Omega>$ for all $x \in M_2$ means that Ω is a common eigenvector for all a_k^*, i.e. $a_k^*\,\Omega = \overline{\omega_k}\,\Omega$ for $k = 1, \ldots, d$. This follows from Lemma 2.4 below. Inserting $\Omega = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ shows that the vector $\Omega_{\mathcal{P}} = \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_d \end{pmatrix} \in \mathcal{P} := \mathbb{C}^d$ is a unit vector satisfying $\Omega_{\mathcal{P}} = a_{11} + a_{21} = a_{12} + a_{22}$. The operator Z is not changed if we apply a unitary in $\mathcal{B}(\mathcal{P})$ to the four vectors $a_{11}, a_{12}, a_{21}, a_{22}$. This follows from the corresponding and well-known non-uniqueness of the operators $\{a_k\}$ in the representation of Z. Thus we can fix an arbitrary two-dimensional plane and realize the vectors $\Omega_{\mathcal{P}}, a_{11}, a_{21}$ there as a right-angled and isosceles triangle. The vectors $\Omega_{\mathcal{P}}, a_{12}, a_{22}$ give another such triangle and in general we need a third dimension to represent the relative position of the two triangles properly. Thus without restriction of generality we can choose $d = 3$ and

$$\Omega_{\mathcal{P}} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad a_{11} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \qquad a_{12} = \frac{1}{2}\begin{pmatrix} 1 \\ c \\ \sqrt{1-|c|^2} \end{pmatrix},$$

$$a_{21} = \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \qquad a_{22} = \frac{1}{2}\begin{pmatrix} 1 \\ -c \\ -\sqrt{1-|c|^2} \end{pmatrix}$$

with a parameter $c \in \mathbb{C}$, $|c| \leq 1$. Thus we have

$$a_1 = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad a_2 = \frac{1}{2}\begin{pmatrix} 1 & c \\ -1 & -c \end{pmatrix}, \quad a_3 = \frac{1}{2}\begin{pmatrix} 0 & \sqrt{1-|c|^2} \\ 0 & -\sqrt{1-|c|^2} \end{pmatrix}.$$

Then the entries of $Z_c(x) = \sum_{k=1}^{3} a_k\, x\, a_k^*$ with parameter $c \in \mathbb{C}$, $|c| \leq 1$ can be computed. We have shown the following result:

Proposition 1.1. *For the stochastic matrix* $\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ *the set of completely positive extensions* $Z : M_2 \to M_2$ *with* $< \Omega, x\Omega > = < \Omega, Z(x)\Omega >$ *for all* $x \in M_2$ *is parametrized by* $c \in \mathbb{C}$, $|c| \leq 1$. *Explicitly: For* $x = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$ *we get* $Z_c(x) =$

$$\frac{1}{4}\begin{pmatrix} 2(x_{11}+x_{22}) + (1+\bar{c})x_{12} + (1+c)x_{21} & (1-\bar{c})x_{12} + (1-c)x_{21} \\ (1-\bar{c})x_{12} + (1-c)x_{21} & 2(x_{11}+x_{22}) + (1+\bar{c})x_{12} + (1+c)x_{21} \end{pmatrix}.$$

Remark 1.2. The Z_c for $|c| \leq 1$ are all different from each other. Note that $Z_{c_1+c_2} = Z_{c_1} + Z_{c_2}$, showing that the convex set of extensions is (affinely) isomorphic to $\{c \in \mathbb{C} : |c| \leq 1\}$. Looking for the minimal number of a_k necessary to represent Z_c, we see that for extremal points ($|c| = 1$) it is 2, otherwise it is 3.

Remark 1.3. The reader may also check that for a matrix $\begin{pmatrix} 1-\lambda & \lambda \\ \mu & 1-\mu \end{pmatrix}$ with $0 < \lambda, \mu < 1$ one can use a similar argument and also gets a parametrization by the unit disc.

2. An Extension Problem (General Case)

Let us define a general setting for the extension problem which has been discussed in section 1 by an example. Suppose $\mathcal{A} \subset \mathcal{B}(\mathcal{G})$ and $\mathcal{B} \subset \mathcal{B}(\mathcal{H})$ are represented von Neumann algebras with cyclic vectors $\Omega_{\mathcal{G}} \in \mathcal{G}$ and $\Omega_{\mathcal{H}} \in \mathcal{H}$. Restricting the corresponding vector states to $\mathcal{A}$ and $\mathcal{B}$ we get normal states $\phi_{\mathcal{A}}$ and $\phi_{\mathcal{B}}$. Then consider a normal unital completely positive map $S : (\mathcal{A}, \phi_{\mathcal{A}}) \to (\mathcal{B}, \phi_{\mathcal{B}})$, i.e. $S : \mathcal{A} \to \mathcal{B}$ and $\phi_{\mathcal{B}} \circ S = \phi_{\mathcal{A}}$.
We are interested in the following set

Definition 2.1.

$$\mathcal{Z}(S, \phi_{\mathcal{B}}) := \begin{array}{l} \{Z : \mathcal{B}(\mathcal{G}) \to \mathcal{B}(\mathcal{H}) \text{ normal unital completely positive and } Z|_{\mathcal{A}} = S \\ \text{and } < \Omega_{\mathcal{G}}, x\Omega_{\mathcal{G}} > = < \Omega_{\mathcal{H}}, Z(x)\Omega_{\mathcal{H}} > \text{ for all } x \in \mathcal{B}(\mathcal{G})\}. \end{array}$$

REMARK 2.2. Note that in section 1 we have computed the set $\mathcal{Z}(S, \phi)$ for S given by a stochastic 2×2-matrix with an invariant state ϕ.

REMARK 2.3. $\mathcal{Z}$ is convex and closed in suitable topologies. For example we can use the topology of pointwise weak*-convergence. It is well known (and easy to check by a Banach-Alaoglu type of argument) that the set of normal unital completely positive maps is compact in this topology (see [**Ar**], 1.2). Thus $\mathcal{Z}$ is the closed convex hull of its extremal points by the Krein-Milman theorem.

Let us note some immediate observations. Using the Stinespring representation ([**Ta**], IV.3.6) and the amplification-induction-theorem ([**Ta**], IV.5.5), we get a representation for $S : \mathcal{B}(\mathcal{G}) \supset \mathcal{A} \to \mathcal{B} \subset \mathcal{B}(\mathcal{H})$ of the form $S(a) = v^* \; a \otimes 1 \; v$, where $a \in \mathcal{A}$, $v : \mathcal{H} \to \mathcal{G} \otimes \mathcal{P}$ is an isometry, $\mathcal{P}$ another Hilbert space. Writing $v\,\xi = \sum_{k=1}^d a_k^*(\xi) \otimes \epsilon_k$ for an ONB $\{\epsilon_k\}$ of $\mathcal{P}$ we have a corresponding decomposition $S(a) = \sum_{k=1}^d a_k\, a\, a_k^*$ for $a \in \mathcal{A}$, $a_k \in \mathcal{B}(\mathcal{G}, \mathcal{H})$, $d = dim\mathcal{P}$. The sum should be interpreted in the strong operator topology if $d = \infty$. Then we have the ansatz

$$Z(x) = v^* \; x \otimes 1 \; v = \sum_{k=1}^d a_k\, x\, a_k^*$$

for all $x \in \mathcal{B}(\mathcal{G})$. This is a normal unital completely positive map extending S. Concerning the states we note:

LEMMA 2.4. *The following assertions are equivalent:*

1. $< \Omega_\mathcal{G}, x\Omega_\mathcal{G} > = < \Omega_\mathcal{H}, Z(x)\Omega_\mathcal{H} >$ *for all* $x \in \mathcal{B}(\mathcal{G})$.
2. *There is a unit vector* $\Omega_\mathcal{P} \in \mathcal{P}$ *such that* $v\Omega_\mathcal{H} = \Omega_\mathcal{G} \otimes \Omega_\mathcal{P}$.
3. *There is a function* $\omega : \{1, \ldots, d\} \to \mathbb{C}$, $k \mapsto \omega_k$ *such that* $a_k^* \, \Omega_\mathcal{H} = \overline{\omega_k}\, \Omega_\mathcal{G}$ *for all* k.

If $\mathcal{G} = \mathcal{H}$ *and* $\Omega_\mathcal{G} = \Omega_\mathcal{H} =: \Omega$, *then* (3) *means that* Ω *is a common eigenvector for all* a_k^*.

PROOF. (2) $\Leftrightarrow$ (3) follows by inserting $\xi = \Omega_\mathcal{H}$ into $v\,\xi = \sum_{k=1}^d a_k^*(\xi) \otimes \epsilon_k$ and observing that $\Omega_\mathcal{P} = \sum_{k=1}^d \overline{\omega_k}\, \epsilon_k$. Further (1) can be written as

$$< \Omega_\mathcal{G}, x\Omega_\mathcal{G} > = < v\,\Omega_\mathcal{H}, \; x \otimes 1 \; v\,\Omega_\mathcal{H} > \quad \text{for all } x \in \mathcal{B}(\mathcal{H}).$$

Thus (2) $\Rightarrow$ (1) is immediate. For the converse assume that $v\,\Omega_\mathcal{H}$ has not the form given in (2). Then inserting $x = p_{\Omega_\mathcal{G}}$, the projection onto $\mathbb{C}\Omega_\mathcal{G}$, yields

$$|< v\,\Omega_\mathcal{H}, p_{\Omega_\mathcal{G}} \otimes 1 \; v\,\Omega_\mathcal{G} >| < 1 \; = < \Omega_\mathcal{G}, p_{\Omega_\mathcal{G}} \Omega_\mathcal{G} >,$$

contradicting (1). □

Thus in an informal language we can restate our extension problem as follows: Can we find Stinespring representations of S such that these additional properties are satisfied? And how many different Z can we construct in this way?

3. A Dilation Problem

We want to relate the extension problem considered in the previous sections to a dilation problem. We start by defining the latter.

Definition 3.1. Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be von Neumann algebras. A normal $*-$homomorphism $j : \mathcal{A} \to \mathcal{B} \otimes \mathcal{C}$ (not necessarily unital) is called a weak tensor dilation (of first order) for a normal unital completely positive map $S : \mathcal{A} \to \mathcal{B}$ if there is a normal state ψ of $\mathcal{C}$ such that $S = P_\psi j$. Here $P_\psi : \mathcal{B} \otimes \mathcal{C} \to \mathcal{B}$ denotes the conditional expectation determined by $P_\psi(b \otimes c) = b\,\psi(c)$.

Remark 3.2. Let us give some comments on this definition. Of course the same definition is possible for C^*-algebras (dropping normality), but our main results concern von Neumann algebras. 'Weak' refers to the fact that the dilation is not assumed to be unital, similar to other weak dilation theories, see [**BP**]. The dilation is called 'tensor' because the conditional expectation used is of tensor type. 'First order' means that higher powers of S are not considered.

Remark 3.3. If $\mathcal{B} = \mathcal{B}(\mathcal{L})$ for a Hilbert space $\mathcal{L}$, then it is easy to construct weak tensor dilations using the Stinespring representation of S. The point of our definition is the inclusion of a tensor product structure into the concept. See the equivalence relation below which makes explicit use of it.

Applying the GNS-construction to $(\mathcal{C}, \psi)$ we get a representation of $\mathcal{C}$ on a Hilbert space $\mathcal{K}$ and a cyclic unit vector $\Omega_\mathcal{K} \in \mathcal{K}$ representing the state ψ. Therefore we do not loose much if in the definition of weak tensor dilations above we replace $(\mathcal{C}, \psi)$ by $(\mathcal{B}(\mathcal{K}), \Omega_\mathcal{K})$.

Suppose now that $\mathcal{A} \subset \mathcal{B}(\mathcal{G})$ and $\mathcal{B} \subset \mathcal{B}(\mathcal{H})$ with cyclic vectors $\Omega_\mathcal{G} \in \mathcal{G}$ and $\Omega_\mathcal{H} \in \mathcal{H}$ yielding states $\phi_\mathcal{A}$ and $\phi_\mathcal{B}$, so that $S : (\mathcal{A}, \phi_\mathcal{A}) \to (\mathcal{B}, \phi_\mathcal{B})$. For a weak tensor dilation j of S we can then define an isometry v associated to j (together with $\phi_\mathcal{B}$):

$$\begin{aligned} v &: \mathcal{G} \to \mathcal{H} \otimes \mathcal{K} \\ a\,\Omega_\mathcal{G} &\mapsto j(a)\,\Omega_\mathcal{H} \otimes \Omega_\mathcal{K}. \end{aligned}$$

The isometric property of v can be seen by the following computation:

$$\| j(a)\,\Omega_\mathcal{H} \otimes \Omega_\mathcal{K} \|^2 = < \Omega_\mathcal{H} \otimes \Omega_\mathcal{H}, j(a^*a)\Omega_\mathcal{H} \otimes \Omega_\mathcal{H} >$$
$$= < \Omega_\mathcal{H}, S(a^*a)\Omega_\mathcal{H} > = < \Omega_\mathcal{G}, a^*a\,\Omega_\mathcal{G} > = \|a\,\Omega_\mathcal{G}\|^2.$$

We can use associated isometries in order to define a natural equivalence relation on weak tensor dilations:

Definition 3.4. Let $\phi_\mathcal{B}$ be faithful. Then we say that two weak tensor dilations j_1, j_2 of S are equivalent if there is a partial isometry $w : \mathcal{K}_1 \to \mathcal{K}_2$ such that $v_2 = (1 \otimes w)v_1$. Here associated objects are given the corresponding subscript.

Remark 3.5. If $\mathcal{B}$ has a separable predual then faithful normal states exist (see [**Sa**], 2.1.9) and Definition 3.4 is applicable. It can be shown that equivalence does not depend on the choice of the faithful state $\phi_\mathcal{B}$, see [**Go**],1.4. Since we do not explicitly need this fact, we omit the proof.

4. Duality

The extension problem and the dilation problem are closely related. Again suppose that $S : (\mathcal{A}, \phi_\mathcal{A}) \to (\mathcal{B}, \phi_\mathcal{B})$ is a normal unital completely positive map and $\mathcal{A} \subset \mathcal{B}(\mathcal{G})$ and $\mathcal{B} \subset \mathcal{B}(\mathcal{H})$ with cyclic vectors $\Omega_\mathcal{G} \in \mathcal{G}$ and $\Omega_\mathcal{H} \in \mathcal{H}$ implementing the states $\phi_\mathcal{A}$ and $\phi_\mathcal{B}$. By restriction of the vector states we also get normal states $\phi_{\mathcal{A}'}$ and $\phi_{\mathcal{B}'}$ on the commutants $\mathcal{A}' \subset \mathcal{B}(\mathcal{G})$ and $\mathcal{B}' \subset \mathcal{B}(\mathcal{H})$. Recall that there exists a unique normal unital completely positive map $S' : (\mathcal{B}', \phi_{\mathcal{B}'}) \to (\mathcal{A}', \phi_{\mathcal{A}'})$ such that $< \Omega_{\mathcal{H}}, S(a)\, b'\, \Omega_{\mathcal{H}} > = < \Omega_{\mathcal{G}}, a\, S'(b')\, \Omega_{\mathcal{G}} >$ for all $a \in \mathcal{A},\ b' \in \mathcal{B}'$ (e.g., use the argument in [**Sa**], 1.21.10). Let us call S' the dual map.

Now let $j : \mathcal{A} \to \mathcal{B} \otimes \mathcal{B}(\mathcal{K})$ be a weak tensor dilation of S, with associated isometry $v : \mathcal{G} \to \mathcal{H} \otimes \mathcal{K}$. Define an operator

$$Z' : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{G}), \quad x \mapsto v^*\, x \otimes 1\, v.$$

Proposition 4.1.

$$Z' \in \mathcal{Z}(S', \phi_{\mathcal{A}'}).$$

Proof. Z' is given in a Stinespring representation, i.e. it is a normal unital completely positive map. The associated isometry $v : a\Omega_{\mathcal{G}} \mapsto j(a)\, \Omega_{\mathcal{H}} \otimes \Omega_{\mathcal{K}}$ satisfies $v\Omega_{\mathcal{G}} = \Omega_{\mathcal{H}} \otimes \Omega_{\mathcal{K}}$ which by Lemma 2.4 implies $< \Omega_{\mathcal{H}}, x\Omega_{\mathcal{H}} > = < \Omega_{\mathcal{G}}, Z'(x)\Omega_{\mathcal{G}} >$ for all $x \in \mathcal{B}(\mathcal{H})$. For all $a_1,\, a_2 \in \mathcal{A}$ and $b' \in \mathcal{B}'$ we get

$$< a_1\Omega_{\mathcal{G}}, v^*\, b' \otimes 1\; v a_2 \Omega_{\mathcal{G}} > = < j(a_1)\Omega_{\mathcal{H}} \otimes \Omega_{\mathcal{K}}, b' \otimes 1\; j(a_2)\, \Omega_{\mathcal{H}} \otimes \Omega_{\mathcal{K}} >$$
$$= < \Omega_{\mathcal{H}} \otimes \Omega_{\mathcal{K}}, j(a_1^* a_2)(b'\Omega_{\mathcal{H}}) \otimes \Omega_{\mathcal{K}} > = < \Omega_{\mathcal{H}}, P_\psi\, j(a_1^* a_2)\, b' \Omega_{\mathcal{H}} >$$
$$= < \Omega_{\mathcal{H}}, S(a_1^* a_2) b' \Omega_{\mathcal{H}} > = < \Omega_{\mathcal{G}}, S'(b') a_1^* a_2 \Omega_{\mathcal{G}} >$$
$$= < a_1 \Omega_{\mathcal{G}}, S'(b') a_2 \Omega_{\mathcal{G}} >, \quad \text{i.e. } v^*\, b' \otimes 1\; v = S'(b').$$

□

Let us look for a kind of converse for the preceding result.

Proposition 4.2. *Given $Z : \mathcal{B}(\mathcal{G}) \to \mathcal{B}(\mathcal{H})$ with $Z \in \mathcal{Z}(S, \phi_{\mathcal{B}})$. Then there exists a weak tensor dilation $j' : \mathcal{B}' \to \mathcal{A}' \otimes \mathcal{B}(\mathcal{P})$ of S' with associated isometry $v' : \mathcal{H} \to \mathcal{G} \otimes \mathcal{P}$ such that $Z(x) = (v')^* x \otimes 1\; v'$ for all $x \in \mathcal{B}(\mathcal{G})$.*

Remark 4.3. Note that j' and v' are not commutants of other objects, but $'$ denotes the connection to the dual map S'.

Proof. By the Stinespring representation theorem we find an isometry $v' : \mathcal{H} \to \mathcal{G} \otimes \mathcal{P}$ such that $Z(x) = (v')^*\; x \otimes 1\; v'$. Because $S = Z|_{\mathcal{A}}$ we also have $S(a) = (v')^*\; a \otimes 1\; v'$ for $a \in \mathcal{A}$. Let p' be the projection from $\mathcal{G} \otimes \mathcal{P}$ onto the minimal part of the Stinespring representation of S, i.e. onto the closure of $(\mathcal{A} \otimes 1)\, v'\mathcal{H}$.

We want to use a result of Arveson ([**Ar**], 1.3, see also [**Ta**], 3.6) on the lifting of commutants. The following version of it is adapted to our needs: If $\mathcal{H}_1$, $\mathcal{H}_2$ are Hilbert spaces, $v : \mathcal{H}_1 \to \mathcal{H}_2$ an isometry, $\mathcal{E} \subset \mathcal{B}(\mathcal{H}_2)$ a selfadjoint algebra with $\overline{\mathcal{E} v \mathcal{H}_1} = \mathcal{H}_2$, then there exists an isomorphism j from $\{v^* \mathcal{E} v\}'$ onto $\mathcal{E}' \cap \{vv^*\}'$ satisfying $j(\cdot) v = v \cdot$ (with $\cdot$ representing elements of $\{v^* \mathcal{E} v\}'$). Note that the last condition determines j uniquely: For $\xi \in \mathcal{H}_1$ and $\epsilon \in \mathcal{E}$ we have

$$j(\cdot)\epsilon v \xi = \epsilon j(\cdot) v \xi = \epsilon v \cdot \xi$$

In fact, if we define j in this way, then it is possible to check the properties stated above. This is done in [**Ar**].

We apply this for $\mathcal{H}_1 = \mathcal{H},\ \mathcal{H}_2 = p'(\mathcal{G} \otimes \mathcal{P})$ and $v = v'$. Define $\mathcal{E} := p'\, \mathcal{A} \otimes 1\, p' = \mathcal{A} \otimes 1\, p'$. Then because $p'(\mathcal{G} \otimes \mathcal{P}) \supset v'\mathcal{H}$ we get

$$\{(v')^* \mathcal{E} v'\}' = \{(v')^*\, \mathcal{A} \otimes 1\, v'\}' = S(\mathcal{A})' \supset \mathcal{B}'$$

and now Arveson's result yields a (normal *-) homomorphism

$$j' : \mathcal{B}' \to \{\mathcal{A} \otimes 1\, p'\}^c,$$

where we have introduced the notation ^{c} to denote the commutant on $\mathcal{H}_2$. For any projection e in a von Neumann algebra $\mathcal{M}$ which is represented on a Hilbert space it is always true that on the range of e the commutant of $e\mathcal{M}e$ equals $\mathcal{M}'e$ (see [**Ta**], 3.10). Here this means that $\{p'\,\mathcal{A}' \otimes \mathcal{B}(\mathcal{P})p'\}^c = (\mathcal{A} \otimes 1)p'$ and thus

$$\{(\mathcal{A} \otimes 1)p'\}^c = (p'\,\mathcal{A}' \otimes \mathcal{B}(\mathcal{P})\,p')^{cc} = p'\,\mathcal{A}' \otimes \mathcal{B}(\mathcal{P})\,p' \subset \mathcal{A}' \otimes \mathcal{B}(\mathcal{P}).$$

We conclude that $j' : \mathcal{B}' \to \mathcal{A}' \otimes \mathcal{B}(\mathcal{P})$ and $j'(1) \le p'$. Applying $j'(\cdot)v' = v'\cdot$ to $\Omega_\mathcal{H}$ gives for $b' \in \mathcal{B}'$

$$v'b'\Omega_\mathcal{H} = j'(b')v'\Omega_\mathcal{H}$$

Because $<\Omega_\mathcal{G}, x\Omega_\mathcal{G}> = <\Omega_\mathcal{H}, Z(x)\Omega_\mathcal{H}>$ for all $x \in \mathcal{B}(\mathcal{G})$ it follows by Lemma 2.4 that there is a unit vector $\Omega_\mathcal{P} \in \mathcal{P}$ such that $v'\Omega_\mathcal{H} = \Omega_\mathcal{G} \otimes \Omega_\mathcal{P}$. Thus

$$v'b'\Omega_\mathcal{H} = j'(b')\,\Omega_\mathcal{G} \otimes \Omega_\mathcal{P},$$

which expresses the fact that v' is the isometry associated to j'. It remains to prove that if ψ is the vector state given by $\Omega_\mathcal{P}$ then $P_\psi j' = S'$.
For $a_1,\, a_2 \in \mathcal{A}$ and $b' \in \mathcal{B}'$ we get

$$\begin{aligned}
&< a_1\Omega_\mathcal{G}, P_\psi j'(b')a_2\Omega_\mathcal{G} > = < (a_1 \otimes 1)\,\Omega_\mathcal{G} \otimes \Omega_\mathcal{P}, (a_2 \otimes 1)j'(b')\,\Omega_\mathcal{G} \otimes \Omega_\mathcal{P} > \\
&= < a_1 \otimes 1\, v'\Omega_\mathcal{H}, a_2 \otimes 1\, v'b'\Omega_\mathcal{H} > = < \Omega_\mathcal{H}, (v')^* a_1^* a_2 \otimes 1\, v'b'\Omega_\mathcal{H} > \\
&= < \Omega_\mathcal{H}, S(a_1^* a_2)b'\Omega_\mathcal{H} > = < \Omega_\mathcal{G}, a_1^* a_2 S'(b')\Omega_\mathcal{G} > \\
&= < a_1\Omega_\mathcal{G}, S'(b')a_2\Omega_\mathcal{G} >, \quad \text{i.e. } P_\psi j' = S'.
\end{aligned}$$

□

The duality between extension and dilation worked out so far takes an especially nice form if we assume that the cyclic vectors $\Omega_\mathcal{G}$ and $\Omega_\mathcal{H}$ are not only cyclic but also separating, i.e. if we consider standard representations. In this case we write

$$T : (\mathcal{A}, \phi_\mathcal{A}) \to (\mathcal{B}, \phi_\mathcal{B}), \quad T' : (\mathcal{B}', \phi_{\mathcal{B}'}) \to (\mathcal{A}', \phi_{\mathcal{A}'})$$

instead of S, S'. It is easy to check that $(T')' = T$, and there is a duality in all statements about algebras and commutants. The distinguished vector states are cyclic for algebras and commutants and we can apply our results with $S = T$ and with $S = T'$.

Theorem 4.4. *There is a bijective correspondence between*

- *weak tensor dilations of T modulo equivalence*
- *elements of $\mathcal{Z}(T', \phi_{\mathcal{A}'})$*

and similar with the roles of T and T' interchanged.

Proof. We get the correspondence by applying Proposition 4.1 to $S = T$ and Proposition 4.2 to $S = T'$. To see that the correspondence is one-to-one consider the Stinespring representation $Z' : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{G}), \quad x \mapsto v^*\, x \otimes 1\, v$ for an extension of T'. Such a representation is determined up to unitary equivalence and this corresponds exactly to the equivalence relation in Definition 3.4 for the weak tensor dilation of T obtained from it. □

To illustrate this result we write down the weak tensor dilations corresponding to the extensions of the stochastic matrix $\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ determined in section 1. Note that in this case we have $\mathcal{A} = \mathcal{A}' = \mathcal{B} = \mathcal{B}'$ and it is easy to check that also $T = T'$. These dilations may be computed using Arveson's lifting as shown in the proof of Proposition 4.2 or they may be guessed from the considerations of section 1. We shorten this process by defining the dilations directly and then we verify the required properties.

Recall from section 1 the vectors $a_{11}, a_{12}, a_{21}, a_{22} \in \mathbb{C}^3$, where a_{12} and a_{22} depend on a parameter $c \in \mathbb{C}$, $\mid c \mid \leq 1$. Let $p_{ij} \in M_3$ be the one-dimensional projection onto $\mathbb{C}\,\overline{a_{ij}}$ (complex conjugation in all components). Now we define

$$\begin{aligned} j_c : \mathbb{C}^2 &\to \mathbb{C}^2 \otimes M_3 \\ b &\mapsto \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes j^{(1)}(b) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes j^{(2)}(b), \end{aligned}$$

with homomorphisms $j^{(1)},\, j^{(2)} : \mathbb{C}^2 \to M_3$ determined by

$$\begin{aligned} j^{(1)}\begin{pmatrix} 1 \\ 0 \end{pmatrix} &= p_{11}, \quad j^{(2)}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = p_{12}, \\ j^{(1)}\begin{pmatrix} 0 \\ 1 \end{pmatrix} &= p_{21}, \quad j^{(2)}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = p_{22}. \end{aligned}$$

From $< a_{11}, a_{21} > = < a_{12}, a_{22} > = 0$ it follows that $j^{(1)}$ and $j^{(2)}$ are embeddings of $\mathbb{C}^2$ into M_3. Note that only the second of these depends on the parameter c. We denote by $\epsilon_1, \epsilon_2, \epsilon_3$ the canonical unit vectors in $\mathcal{K} = \mathbb{C}^3$, in particular we fix $\Omega_{\mathcal{K}} := \epsilon_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. For all $i, j = 1, 2$ we find that $p_{ij}\Omega_{\mathcal{K}} = \overline{a_{ij}} = \sum_{k=1}^3 (a_k^*)_{ji}\, \epsilon_k$. Using this we can verify immediately that for all c the homomorphism j_c together with the vector state given by $\Omega_{\mathcal{K}}$ provides us with a weak tensor dilation for the map T corresponding to the stochastic matrix. Further for $b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \in \mathbb{C}^2$

$$v_c\, b\, \Omega := j_c(b)\, \Omega \otimes \Omega_{\mathcal{K}} = \frac{1}{\sqrt{2}} \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes j^{(1)}(b)\, \Omega_{\mathcal{K}} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes j^{(2)}(b)\, \Omega_{\mathcal{K}} \right)$$

$$= \frac{1}{\sqrt{2}} \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes (b_1 p_{11} + b_2 p_{21})\, \Omega_{\mathcal{K}} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes (b_1 p_{12} + b_2 p_{22})\, \Omega_{\mathcal{K}} \right)$$

$$= \sum_{k=1}^{3} \frac{1}{\sqrt{2}} \left((a_k^*)_{11} \begin{pmatrix} b_1 \\ 0 \end{pmatrix} + (a_k^*)_{12} \begin{pmatrix} b_2 \\ 0 \end{pmatrix} + (a_k^*)_{21} \begin{pmatrix} 0 \\ b_1 \end{pmatrix} + (a_k^*)_{22} \begin{pmatrix} 0 \\ b_2 \end{pmatrix} \right) \otimes \epsilon_k$$

$$= \sum_{k=1}^{3} a_k^*(b\Omega) \otimes \epsilon_k.$$

We conclude that

$$v_c \xi = \sum_{k=1}^{3} a_k^*(\xi) \otimes \epsilon_k \quad \textit{for all } \xi \in \mathbb{C}^2,$$

which shows that the isometry v_c associated to j_c is also the isometry occurring in the Stinespring representation of $Z_c = \sum_{k=1}^3 a_k \cdot a_k^*$. Therefore the dilations j_c and the extensions Z_c from section 1 correspond to each other in the way described by Theorem 4.4.

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