Functional Models and Minimal Contractive Liftings

Rolf Gohm
Department of Mathematics and Physics
Aberystwyth University

.

Joint work with S. Dey, K.J. Haria (IIT Bombay)
preprint see arXiv:1402.4220

IWOTA Amsterdam
July 14-18, 2014

Minimal contractive liftings

Let C be a contraction on a Hilbert space $\mathcal{H}_C$.

$$E = \begin{pmatrix} C & 0 \\ B & A \end{pmatrix}$$

on the Hilbert space $\mathcal{H}_E = \mathcal{H}_C \oplus \mathcal{H}_A$ is called a **lifting** of C.

The lifling E is called **contractive** if E is a contraction.

The lifting E is called **minimal** if $\mathcal{H}_E$ is the smallest E-invariant subspace containing $\mathcal{H}_C$.

Minimal contractive liftings

Let C be a contraction on a Hilbert space $\mathcal{H}_C$.

$$E = \begin{pmatrix} C & 0 \\ B & A \end{pmatrix}$$

on the Hilbert space $\mathcal{H}_E = \mathcal{H}_C \oplus \mathcal{H}_A$ is called a **lifting** of C.

The lifling E is called **contractive** if E is a contraction.

The lifting E is called **minimal** if $\mathcal{H}_E$ is the smallest E-invariant subspace containing $\mathcal{H}_C$.

Example: the minimal isometric dilation

Minimal contractive liftings

Let C be a contraction on a Hilbert space $\mathcal{H}_C$.

$$E = \begin{pmatrix} C & 0 \\ B & A \end{pmatrix}$$

on the Hilbert space $\mathcal{H}_E = \mathcal{H}_C \oplus \mathcal{H}_A$ is called a **lifting** of C.

The lifling E is called **contractive** if E is a contraction.

The lifting E is called **minimal** if $\mathcal{H}_E$ is the smallest E-invariant subspace containing $\mathcal{H}_C$.

Example: the minimal isometric dilation

Two liftings E and E' of C are called **unitarily equivalent** if there is an intertwining unitary which restricts to the identity on $\mathcal{H}_C$.

Minimal contractive liftings

Let C be a contraction on a Hilbert space $\mathcal{H}_C$.

$$E = \begin{pmatrix} C & 0 \\ B & A \end{pmatrix}$$

on the Hilbert space $\mathcal{H}_E = \mathcal{H}_C \oplus \mathcal{H}_A$ is called a **lifting** of C.

The lifling E is called **contractive** if E is a contraction.

The lifting E is called **minimal** if $\mathcal{H}_E$ is the smallest E-invariant subspace containing $\mathcal{H}_C$.

Example: the minimal isometric dilation

Two liftings E and E' of C are called **unitarily equivalent** if there is an intertwining unitary which restricts to the identity on $\mathcal{H}_C$.

We give a **classification** of (unitary equivalence classes of) minimal contractive liftings by **characteristic functions**.

Main Theorem (for single contractions)

Given a contraction C on a Hilbert space $\mathcal{H}_C$.

Then we have a one-to-one correspondence between

- **minimal contractive liftings E of C**
  (up to unitary equivalence)

- $\mathcal{B}(\mathcal{D}, \mathcal{D}_C)$**-valued Schur functions Θ on the unit disk**
  (up to a unitary on $\mathcal{D}$)
  with the following **injectivity** property:
  if $0 \neq \delta \in \mathcal{D}$ then $z \mapsto \Theta(z)\delta$ is not the zero function

Here $\dim \mathcal{D}$ is equal to the defect $\dim \mathcal{D}_E$ of E.

Main Theorem (for single contractions)

Given a contraction C on a Hilbert space $\mathcal{H}_C$.

Then we have a one-to-one correspondence between

- **minimal contractive liftings E of C**
  (up to unitary equivalence)

- $\mathcal{B}(\mathcal{D}, \mathcal{D}_C)$**-valued Schur functions Θ on the unit disk**
  (up to a unitary on $\mathcal{D}$)
  with the following **injectivity** property:
  if $0 \neq \delta \in \mathcal{D}$ then $z \mapsto \Theta(z)\delta$ is not the zero function

Here $\dim \mathcal{D}$ is equal to the defect $\dim \mathcal{D}_E$ of E.

We call Θ the **characteristic function of the lifting E**.

Instructive special case

Given a contraction C on a Hilbert space $\mathcal{H}_C$ with 1-dimensional defect.

Instructive special case

Given a contraction C on a Hilbert space $\mathcal{H}_C$
with 1-dimensional defect.

Then we have a one-to-one correspondence between

- **minimal contractive liftings E of C**
  with 1-dimensional defect
  (up to unitary equivalence)
- **non-zero ($\mathbb{C}$-valued) Schur functions on the unit disk**
  (up to unimodular factor)

Example

For the contraction $C = \frac{1}{2}$ on $\mathcal{H}_C = \mathbb{C}$
we have the minimal contractive liftings

$$E_\alpha = \begin{pmatrix} \dfrac{1}{2} & 0 \\ \dfrac{\sqrt{3}}{2}(1-|\alpha|^2)^{\frac{1}{2}} & \alpha \end{pmatrix}, \quad |\alpha| < 1$$

(all with 1-dimensional defect).

Example

For the contraction $C = \frac{1}{2}$ on $\mathcal{H}_C = \mathbb{C}$ we have the minimal contractive liftings

$$E_\alpha = \begin{pmatrix} \dfrac{1}{2} & 0 \\ \dfrac{\sqrt{3}}{2}(1-|\alpha|^2)^{\frac{1}{2}} & \alpha \end{pmatrix}, \quad |\alpha| < 1$$

(all with 1-dimensional defect).

The corresponding characteristic functions are the Möbius transformations

$$\Theta_\alpha(z) = \frac{z - \alpha}{1 - \bar{\alpha} z} \ .$$

Main Theorem (for row contractions)

The result can be generalized to row contractions,
with Schur functions replaced by multi-analytic operators.
We state the result (but don't explain the terminology).

Main Theorem (for row contractions)

The result can be generalized to row contractions,
with Schur functions replaced by multi-analytic operators.
We state the result (but don't explain the terminology).

Theorem:
Given: a row contraction $\underline{C} = (C_1, \dots C_d)$
on a Hilbert space $\mathcal{H}_C$
Then we have a one-to-one correspondence between

- **minimal contractive liftings** $\underline{E} = (E_1, \dots, E_d)$ **of** $\underline{C}$
(up to unitary equivalence)
- **multi-analytic operators with injective symbols**
$\Theta : \mathcal{D} \to \Gamma \otimes \mathcal{D}_C$ (where Γ is the full Fock space over $\mathbb{C}^d$)
(up to a unitary on $\mathcal{D}$)

Here $\dim \mathcal{D}$ is equal to the defect $\dim \mathcal{D}_E$ of $\underline{E}$.
In this case we call the multi-analytic operator or its symbol the **characteristic function of the lifting** E.

idea of proof(1): functional model

- Given: $\mathcal{B}(\mathcal{D},\mathcal{L})$-valued Schur function Θ on the unit disk.
  multiplication operator $M_\Theta : H^2(\mathcal{D}) \to H^2(\mathcal{L})$
  model space: $\mathcal{H}_\Theta := H^2(\mathcal{L}) \oplus clos[\sqrt{I - M_\Theta^* M_\Theta}\, H^2(\mathcal{D})]$
  isometric embedding: $$\begin{aligned} W_\Theta : H^2(\mathcal{D}) &\to \mathcal{H}_\Theta \\ f &\mapsto M_\Theta f \oplus \sqrt{I - M_\Theta^* M_\Theta}\, f \end{aligned}$$
  By this construction we can realize the multiplication operator M_Θ as a restriction of a projection (to the first component).

idea of proof(1): functional model

- Given: $\mathcal{B}(\mathcal{D},\mathcal{L})$-valued Schur function Θ on the unit disk.
  multiplication operator $M_\Theta : H^2(\mathcal{D}) \to H^2(\mathcal{L})$
  model space: $\mathcal{H}_\Theta := H^2(\mathcal{L}) \oplus clos[\sqrt{I - M_\Theta^* M_\Theta}\, H^2(\mathcal{D})]$
  isometric embedding: $W_\Theta : H^2(\mathcal{D}) \to \mathcal{H}_\Theta$
  $$f \mapsto M_\Theta f \oplus \sqrt{I - M_\Theta^* M_\Theta}\, f$$
  By this construction we can realize the multiplication operator M_Θ as a restriction of a projection (to the first component).
- traditional use:
  functional model for a contraction A (Sz.-Nagy, Foias)
  $V := M_z \oplus \sqrt{I - M_\Theta^* M_\Theta}\, M_z$ (isometry on $\mathcal{H}_\Theta$)
  invariant subspace $W_\Theta H^2(\mathcal{D})$
  co-invariant subspace $\mathcal{H}_A := \mathcal{H}_\Theta \ominus W_\Theta H^2(\mathcal{D})$
  contraction $A := P_{\mathcal{H}_A} V|_{\mathcal{H}_A}$ (completely non-coisometric)

idea of proof(2): the map $\mathcal{E}$: Schur functions to liftings

Given: a contraction C on a Hilbert space $\mathcal{H}_C$ and
a $\mathcal{B}(\mathcal{D}, \mathcal{D}_C)$-valued Schur function Θ on the unit disk.

idea of proof(2): the map $\mathcal{E}$: Schur functions to liftings

Given: a contraction C on a Hilbert space $\mathcal{H}_C$ and a $\mathcal{B}(\mathcal{D}, \mathcal{D}_C)$-valued Schur function Θ on the unit disk.

Hilbert space:
$\hat{\mathcal{H}} := \mathcal{H}_C \oplus \mathcal{H}_\Theta = \mathcal{H}_C \oplus H^2(\mathcal{D}_C) \oplus clos[\sqrt{I - M_\Theta^* M_\Theta}\, H^2(\mathcal{D})]$
On $\mathcal{H}_C \oplus H^2(\mathcal{D}_C)$ we have the minimal isometric dilation of C.
On $\mathcal{H}_\Theta$ we have the isometry V from the functional model.
They fit together to form an isometry $\hat{V}$ on $\hat{\mathcal{H}}$.

idea of proof(2): the map $\mathcal{E}$: Schur functions to liftings

Given: a contraction C on a Hilbert space $\mathcal{H}_C$ and
a $\mathcal{B}(\mathcal{D}, \mathcal{D}_C)$-valued Schur function Θ on the unit disk.

Hilbert space:
$\hat{\mathcal{H}} := \mathcal{H}_C \oplus \mathcal{H}_\Theta = \mathcal{H}_C \oplus H^2(\mathcal{D}_C) \oplus clos[\sqrt{I - M_\Theta^* M_\Theta}\, H^2(\mathcal{D})]$
On $\mathcal{H}_C \oplus H^2(\mathcal{D}_C)$ we have the minimal isometric dilation of C.
On $\mathcal{H}_\Theta$ we have the isometry V from the functional model.
They fit together to form an isometry $\hat{V}$ on $\hat{\mathcal{H}}$.

$\hat{\mathcal{H}} = \mathcal{H}_C \oplus \mathcal{H}_\Theta = \mathcal{H}_C \oplus \mathcal{H}_A \oplus W_\Theta H^2(\mathcal{D}) = \mathcal{H}_E \oplus W_\Theta H^2(\mathcal{D})$
with $\mathcal{H}_E := \mathcal{H}_C \oplus \mathcal{H}_A$

$$E = E_{C,\Theta} := P_{\mathcal{H}_E} \hat{V}|_{\mathcal{H}_E} = \begin{pmatrix} C & 0 \\ B & A \end{pmatrix}$$

is a contractive lifting of C.

idea of proof(2): the map $\mathcal{E}$: Schur functions to liftings

Given: a contraction C on a Hilbert space $\mathcal{H}_C$ and a $\mathcal{B}(\mathcal{D}, \mathcal{D}_C)$-valued Schur function Θ on the unit disk.

Hilbert space:
$\hat{\mathcal{H}} := \mathcal{H}_C \oplus \mathcal{H}_\Theta = \mathcal{H}_C \oplus H^2(\mathcal{D}_C) \oplus clos[\sqrt{I - M_\Theta^* M_\Theta}\, H^2(\mathcal{D})]$
On $\mathcal{H}_C \oplus H^2(\mathcal{D}_C)$ we have the minimal isometric dilation of C.
On $\mathcal{H}_\Theta$ we have the isometry V from the functional model.
They fit together to form an isometry $\hat{V}$ on $\hat{\mathcal{H}}$.

$\hat{\mathcal{H}} = \mathcal{H}_C \oplus \mathcal{H}_\Theta = \mathcal{H}_C \oplus \mathcal{H}_A \oplus W_\Theta H^2(\mathcal{D}) = \mathcal{H}_E \oplus W_\Theta H^2(\mathcal{D})$
with $\mathcal{H}_E := \mathcal{H}_C \oplus \mathcal{H}_A$

$$E = E_{C,\Theta} := P_{\mathcal{H}_E} \hat{V}|_{\mathcal{H}_E} = \begin{pmatrix} C & 0 \\ B & A \end{pmatrix}$$

is a contractive lifting of C.

- Given C, this provides a **map** $\mathcal{E}$ from $\mathcal{B}(\mathcal{D}, \mathcal{D}_C)$-valued Schur functions Θ to contractive liftings $\mathcal{E}(\Theta) := E_{C,\Theta}$ of C.

idea of proof(3): the map $\mathcal{M}$: liftings to Schur functions

Given: a contraction C on a Hilbert space $\mathcal{H}_C$
and a contractive lifting E of C.

idea of proof(3): the map $\mathcal{M}$: liftings to Schur functions

Given: a contraction C on a Hilbert space $\mathcal{H}_C$
and a contractive lifting E of C.

Then the minimal isometric dilation of E,
realized on $\mathcal{H}_E \oplus H^2(\mathcal{D}_E)$,
can be restricted to a minimal isometric dilation of C,
unitarily equivalent to a realization on $\mathcal{H}_C \oplus H^2(\mathcal{D}_C)$.

idea of proof(3): the map $\mathcal{M}$: liftings to Schur functions

Given: a contraction C on a Hilbert space $\mathcal{H}_C$
and a contractive lifting E of C.

Then the minimal isometric dilation of E,
realized on $\mathcal{H}_E \oplus H^2(\mathcal{D}_E)$,
can be restricted to a minimal isometric dilation of C,
unitarily equivalent to a realization on $\mathcal{H}_C \oplus H^2(\mathcal{D}_C)$.

The orthogonal projection onto $H^2(\mathcal{D}_C)$ restricted to $H^2(\mathcal{D}_E)$
gives us a multiplication operator $M_{C,E}$.
Its symbol is a $\mathcal{B}(\mathcal{D}_E, \mathcal{D}_C)$-valued Schur function $\Theta_{C,E}$.

idea of proof(3): the map $\mathcal{M}$: liftings to Schur functions

Given: a contraction C on a Hilbert space $\mathcal{H}_C$
and a contractive lifting E of C.

Then the minimal isometric dilation of E,
realized on $\mathcal{H}_E \oplus H^2(\mathcal{D}_E)$,
can be restricted to a minimal isometric dilation of C,
unitarily equivalent to a realization on $\mathcal{H}_C \oplus H^2(\mathcal{D}_C)$.

The orthogonal projection onto $H^2(\mathcal{D}_C)$ restricted to $H^2(\mathcal{D}_E)$
gives us a multiplication operator $M_{C,E}$.
Its symbol is a $\mathcal{B}(\mathcal{D}_E, \mathcal{D}_C)$-valued Schur function $\Theta_{C,E}$.

- Given C, this provides a **map** $\mathcal{M}$ from contractive liftings E of C to multiplication operators $\mathcal{M}(E) = \mathrm{M}_{C,E}$.

idea of proof(4): the correspondence

The proof is completed by the following observations:

- $\mathcal{E}(M)$ is always **minimal**.

idea of proof(4): the correspondence

The proof is completed by the following observations:

- $\mathcal{E}(M)$ is always **minimal**.
- If E is a minimal contractive lifting
  then $\mathcal{M}(E)$ has an **injective** symbol Θ.

idea of proof(4): the correspondence

The proof is completed by the following observations:

- $\mathcal{E}(M)$ is always **minimal**.
- If E is a minimal contractive lifting
  then $\mathcal{M}(E)$ has an **injective** symbol Θ.
- If M_Θ has an injective symbol Θ then $\mathcal{M} \circ \mathcal{E}(M_\Theta) = M_\Theta$.

  In particular for each injective symbol Θ there is a minimal contractive lifting E such that $\mathcal{M}(E)$ has symbol Θ.

idea of proof(4): the correspondence

The proof is completed by the following observations:

- $\mathcal{E}(M)$ is always **minimal**.
- If E is a minimal contractive lifting then $\mathcal{M}(E)$ has an **injective** symbol Θ.
- If M_Θ has an injective symbol Θ then $\mathcal{M} \circ \mathcal{E}(M_\Theta) = M_\Theta$.

  In particular for each injective symbol Θ there is a minimal contractive lifting E such that $\mathcal{M}(E)$ has symbol Θ.
- Minimal contractive liftings E and E' are unitarily equivalent if and only if $\mathcal{M}(E)$ and $\mathcal{M}(E')$ are equivalent (i.e., the same up to a unitary $\mathcal{D}_E \to \mathcal{D}_{E'}$).

idea of proof(4): the correspondence

The proof is completed by the following observations:

- $\mathcal{E}(M)$ is always **minimal**.
- If E is a minimal contractive lifting
  then $\mathcal{M}(E)$ has an **injective** symbol Θ.
- If M_Θ has an injective symbol Θ then $\mathcal{M} \circ \mathcal{E}(M_\Theta) = M_\Theta$.

  In particular for each injective symbol Θ there is a minimal contractive lifting E such that $\mathcal{M}(E)$ has symbol Θ.
- Minimal contractive liftings E and E' are unitarily equivalent
  if and only if $\mathcal{M}(E)$ and $\mathcal{M}(E')$ are equivalent
  (i.e., the same up to a unitary $\mathcal{D}_E \to \mathcal{D}_{E'}$).
- Conclusion: On (unitary) equivalence classes
  the mappings $\mathcal{E}$ and $\mathcal{M}$ are well defined
  and inverse to each other.

  This proves the theorem. Everything is constructive!

Characteristic function of lower right corner A

If $M_\Theta : H^2(\mathcal{D}) \to H^2(\mathcal{L})$ is also

- purely contractive: $\|P_{\mathcal{L}} M_\Theta \delta\| < \|\delta\|$ for all $0 \neq \delta \in \mathcal{D}$
- Szegö condition:

$$clos[(I - M_\Theta^* M_\Theta)^{\frac{1}{2}} H^2(\mathcal{D})] = clos[(I - M_\Theta^* M_\Theta)^{\frac{1}{2}} (H^2(\mathcal{D}) \ominus \mathcal{D})]$$

then M_Θ as a characteristic function of the lifting $\begin{pmatrix} C & 0 \\ B & A \end{pmatrix}$ (if $\mathcal{L} = \mathcal{D}_C$) is also the characteristic function of the lower right corner A in the sense of Sz.-Nagy and Foias (or Popescu for row contractions).

But otherwise the two characteristic functions are different.

An application: factorization of liftings

The correspondence shown above allows us to investigate minimal contractive liftings by looking at the symbols of their characteristic functions. We finish with an example of such an application.

An application: factorization of liftings

The correspondence shown above allows us to investigate minimal contractive liftings by looking at the symbols of their characteristic functions. We finish with an example of such an application.

For iterated liftings:

$$E = \begin{pmatrix} C & 0 \\ B & A \end{pmatrix}$$

$$E' = \begin{pmatrix} E & 0 \\ B' & A' \end{pmatrix}$$

An application: factorization of liftings

The correspondence shown above allows us to investigate minimal contractive liftings by looking at the symbols of their characteristic functions. We finish with an example of such an application.

For iterated liftings:

$$E = \begin{pmatrix} C & 0 \\ B & A \end{pmatrix}$$

$$E' = \begin{pmatrix} E & 0 \\ B' & A' \end{pmatrix}$$

we find a factorization of the characteristic functions:

$$M_{C,E'} = M_{C,E} \, M_{E,E'} \, .$$

An application: factorization of liftings

The correspondence shown above allows us to investigate minimal contractive liftings by looking at the symbols of their characteristic functions. We finish with an example of such an application.
For iterated liftings:

$$E = \begin{pmatrix} C & 0 \\ B & A \end{pmatrix}$$

$$E' = \begin{pmatrix} E & 0 \\ B' & A' \end{pmatrix}$$

we find a factorization of the characteristic functions:

$$M_{C,E'} = M_{C,E} \; M_{E,E'} \, .$$

Converse is also true: any factorization of $M_{C,E'}$ into two functions with injective symbols implies the existence of an intermediate lifting E.