Repeated Interactions: Operator Theory meets Quantum Physics

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Operator Theory

An operator T on a Hilbert space $\mathcal{H}$ is **selfadjoint** if

$$\langle Ty, x\rangle = \langle y, Tx\rangle \, .$$

The spectrum $\sigma(T)$ of a selfadjoint operator is contained in $\mathbb{R}$.

For $A \subset \sigma(T)$ there exists a **spectral projection** $E_A(T)$.

Spectral projections $E_A(T)$ and $E_B(T)$ are orthogonal to each other if A and B are disjoint and they sum up to the identity operator if the whole spectrum $\sigma(T)$ is covered.

In finite dimension they correspond to (sums of) eigenspaces.

Quantum Physics

A quantum system is described by a Hilbert space $\mathcal{H}$ and any unit vector $\psi \in \mathcal{H}$ describes a **state** of the system.

A selfadjoint operator T on $\mathcal{H}$ is interpreted as an **observable** quantity: The number

$$\langle \psi, E_A(T)\, \psi \rangle$$

is interpreted as the **probability** to get a value inside the set A if the system is in the state ψ and the observable quantity T is observed.

Dynamics

Evolutions in time are described by **unitaries** U:

$$\psi \mapsto U\psi$$

Combined systems are described by a **tensor product** of Hilbert spaces

$$\mathcal{H} \otimes \mathcal{K}\,.$$

So to describe the evolution of a combined system we need to specify a unitary U on $\mathcal{H} \otimes \mathcal{K}$.

In this case U is also called an **interaction**.

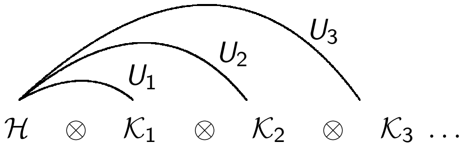
Repeated Interactions

Very interesting problems (mathematical and physical!) arise if we let the system $\mathcal{H}$ **successively interact** with many **independent** copies of the system $\mathcal{K}$:

$$\mathcal{K}_1,\ \mathcal{K}_2,\ \mathcal{K}_3, \ldots, \mathcal{K}_n$$

by using copies $U_1, U_2, U_3, \ldots, U_n$ of the interaction U.
After n steps we have a **dynamics**

$$U_n \ldots U_2 U_1 : \mathcal{H} \otimes \mathcal{K}_1 \otimes \ldots \otimes \mathcal{K}_n \rightarrow \mathcal{H} \otimes \mathcal{K}_1 \otimes \ldots \otimes \mathcal{K}_n$$



Interesting problems

Let the interactions take place! Good questions to ask:

Filtering Problem: By systematically measuring observables on the copies $\mathcal{K}_i$ after the interactions, can we use the results to predict the outcomes of further measurements?
What can we conclude from such results about the system $\mathcal{H}$?

Preparability Problem: Can we prepare a specified state on $\mathcal{H}$ by repeated interactions if we start with known states on the $\mathcal{K}_i$?

How does all this depend on the unitary U?

Quantum Control is considered to be a major component of future technologies. Our setting considered here is a nice playground for mathematicians (**operator theory**) and physicists (**quantum theory**) to come together.

An Experiment. Nobel Prize 2012: S. Haroche

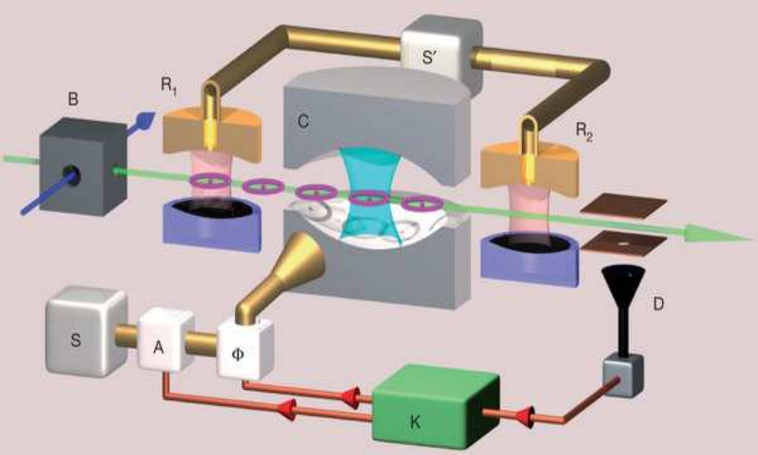


Figure:
Real-time quantum feedback prepares and stabilizes photon number states. Nature 477, 73-77 (2011)