Asymptotic Completeness and Controllability of Open Quantum Systems

Rolf Gohm
Department of Mathematics
Aberystwyth University

Institut Henri Poincaré, Paris
Workshop Operator Algebras and Quantum Information Theory
11-15 September 2017

,

Plan of the talk

Abstract: Repeated interactions of an open quantum system with copies of another system can be interpreted as a quantum Markov process. The scattering theory concept of asymptotic completeness is closely related to the preparability of states and hence to the controllability of the open system. We state and explain the main results and examples (micromaser) in

R. Gohm, F.Haag, B.Kümmerer, Universal Preparability and Asymptotic Completeness, CMP 352(1) (2017), 59-94

We discuss the operator algebraic techniques developed for the proofs and reflect about their potential.

Repeated Interactions

- $\mathcal{A}, \mathcal{C}$ von Neumann algebras,
  describing observables of physical systems

  $J : \mathcal{A} \to \mathcal{A} \otimes \mathcal{C}$ normal $*$-homomorphism, called a **transition**, describing an interaction between the two systems
  (for example $J(a) = u^* \, a \otimes \mathbb{1} \, u$ with a unitary $u \in \mathcal{A} \otimes \mathcal{C}$ specifying a quantum Schrödinger dynamics of the combined system)

- We want to study an **interaction of $\mathcal{A}$ with a sequence of copies $\mathcal{C}_{(1)}, \mathcal{C}_{(2)}, \ldots$ of $\mathcal{C}$.** After n steps

$$J_n : \mathcal{A} \to \mathcal{A} \otimes \bigotimes_{i=1}^{n} \mathcal{C}_{(i)}, \; J_n = J_{(1)} J_{(2)} \ldots J_{(n)}$$

  with $\quad J_{(i)} : \mathcal{A} \to \mathcal{A} \otimes \mathcal{C}_{(i)} \quad$ a copy of $J : \mathcal{A} \to \mathcal{A} \otimes \mathcal{C}$.

  (for example a sequence of atoms interacting with an electromagnetic field in micromaser experiments, see later)

State preparation by repeated interactions

- We consider the transition J as given and we want, by arranging the $\mathcal{C}_{(i)}$ suitably, influence the system $\mathcal{A}$. This produces natural **questions of a control theoretic flavour**.
- Let σ, ρ be normal states of $\mathcal{A}$. We define

  ρ **is J-preparable from** σ if there exists $(n_k) \subset \mathbb{N}$ and normal states θ_k of $\bigotimes_{i=1}^{n_k} \mathcal{C}_{(i)}$ such that $(\sigma \otimes \theta_k)J_{n_k}(a)$ converges to $\rho(a)$ for all $a \in \mathcal{A}$.

  In words: **If we are able to prepare the states θ_k then, by running the interaction J repeatedly, we are able to change the state of $\mathcal{A}$ from σ to ρ, to any precision required.**
- We say that ρ is **universally J-preparable** if we can find (θ_k) which work **simultaneously for all initial states σ**.
  This is a very attractive situation because we don't need to know the initial state and we are still able to prepare ρ.

A time reversal argument

- The following argument shows that universal preparability is not so uncommon than one might think at first sight.

  Suppose $J(a) = \alpha(a \otimes \mathbb{1})$ with a $*$-automorphism of $\mathcal{A} \otimes \mathcal{C}$. Then we have a time reversed version $J^r(a) = \alpha^{-1}(a \otimes \mathbb{1})$ and

  Theorem.
  Suppose $J : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}) \otimes \mathcal{C}$. If there exists a vector state ω_ξ which is both J- and J^r-universally preparable then all normal states are universally J-preparable.

- Main idea: Given ρ to be prepared, check that if a vector state ω_ξ is J^r-preparable from ρ then ρ is J-preparable from ω_ξ. Hence from an arbitrary initial state we first prepare ω_ξ (using universal J-preparability) and from there we prepare ρ (using universal J^r-preparability and the argument above).

Example: Generalized Micromaser

$$\mathcal{A} = \mathcal{B}(\ell^2(\mathbb{N}_0)) \qquad \text{one mode of a field}$$
$$\mathcal{C} = \mathcal{B}(\mathbb{C}^2) = M_2 \quad \text{two-level-atom}$$

Bases $|0\rangle, |1\rangle, |2\rangle, \ldots$ resp. $|0\rangle, |1\rangle$, $\quad J(a) = u^* \, a \otimes \mathbb{1} \, u$ with

$$\begin{aligned} u\,|00\rangle &= |00\rangle, \quad \text{and for } n \geq 1 \\ u\,|n-1,1\rangle &= \alpha_n^+ \,|n-1,1\rangle + \beta_n \,|n,0\rangle \\ u\,|n,0\rangle &= \beta_n^+ \,|n-1,1\rangle + \alpha_n \,|n,0\rangle \end{aligned}$$

with unitary 2×2-matrices $\begin{pmatrix} \alpha_n^+ & \beta_n^+ \\ \beta_n & \alpha_n \end{pmatrix}$.

- Hence if the total energy is $n > 0$ then one quantum of energy is exchanged with probability $|\beta_n|^2 = |\beta_n^+|^2$ (or not with prob. $|\alpha_n|^2 = |\alpha_n^+|^2$). With specific coefficients this can be obtained by **discretization of the Jaynes-Cummings model in quantum optics**. We call it a **generalized micromaser**.

Generalized Micromaser: Results

- **Theorem.** *In the generalized micromaser all normal states are universally J-preparable iff for all $n \geq 1 \quad \beta_n \neq 0$ (no trapping states).*
- Idea of proof: We apply the previous result for the vector state $|0\rangle\langle 0|$ (ground state of the field). It is intuitively plausible and can be checked that this ground state is both J- and J^r-universally preparable by sending in sufficiently many atoms in their ground state which suck out more and more energy from the field.
- Simulations seem to indicate that this procedure often is also the most efficient one. I am not aware of any rigorous proofs in this direction.

Noncommutative Markov processes, Stationarity

- The general question: Which interactions provide universal preparability of states?
- We discuss this in the following setting. Suppose there are faithful normal states φ on $\mathcal{A}$ and ψ on $\mathcal{C}$ such that

$$(\varphi \otimes \psi) \circ J = \varphi \,.$$

  We can think of this as a **stationary noncommutative Markov process**. Think of $T_\psi = P_\psi J : \mathcal{A} \to \mathcal{A}$, with conditional expectation $P_\psi(a \otimes c) = a\,\psi(c)$, as the **transition operator** of this noncommutative Markov process.

  In fact $(T_\psi)^n = P_{\bigotimes_1^n \psi} J_n$, which is the typical semigroup property of the transition operator of a Markov process. Then the condition above translates into $\varphi \circ T_\psi = \varphi$, i.e., φ is a **stationary faithful normal state**.

Asymptotic Completeness

- For stationary Markov processes recall the notion of **asymptotic completeness**, first introduced in: *B. Kümmerer, H. Maassen, A Scattering Theory for Markov Chains. IDAQP 3 (2000), 161-176*

  TFAE:

  1. J is asymptotically complete.
  2. There exists a vNA-isomorphism $\mathcal{A} \otimes \bigotimes_1^\infty \mathcal{C} \to \bigotimes_1^\infty \mathcal{C}$ which intertwines the J-induced dynamics and the tensor shift. (the isomorphism is the Møller operator of this scattering theory)
  3. $\| Q_n J_n(a) - J_n(a) \|_{\varphi \otimes \bigotimes_1^n \psi} \to 0$ for all $a \in \mathcal{A}$, with conditional expectation $Q_n(a \otimes c_n) = \varphi(a) \, \mathbb{1} \otimes c_n$

Connection to Preparability

- Roughly, asymptotic completeness means that, in the Heisenberg picture, the observables

$$a \in \mathcal{A} \quad \simeq \quad a \otimes \mathbb{1} \in \mathcal{A} \otimes \bigotimes_{1}^{\infty} \mathcal{C}$$

develop asymptotically into something of the form

$$\mathbb{1} \otimes c_\infty \in \mathcal{A} \otimes \bigotimes_{1}^{\infty} \mathcal{C}.$$

- Hence in the Schrödinger picture, if the original state is $\sigma \otimes \psi_\infty$ then asymptotically we get the expectation values

$$\sigma \otimes \psi_\infty(\mathbb{1} \otimes c_\infty) = \psi_\infty(c_\infty),$$

independent of σ.
So we should get universal preparability for all (normal) states from asymptotic completeness.

Asymptotic Completeness and Universal Preparability

- Modulo technicalities this is true:

  Theorem.
  Suppose that for a transition $J : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}) \otimes \mathcal{C}$ there exist faithful normal states φ on $\mathcal{B}(\mathcal{H})$ and ψ on $\mathcal{C}$ which yield a stationary Markov process. TFAE
  - *J is asymptotically complete.*
  - *All normal states of $\mathcal{B}(\mathcal{H})$ are universally J-preparable and J is tight.*
- Tightness is a technical condition (see later) which in many cases is satisfied automatically. For example if $\mathcal{A} = \mathcal{B}(\mathcal{H})$ and $J(a) = \alpha(a \otimes \mathbb{1})$ with an automorphism α such that $\varphi \otimes \psi$ is α-invariant then tightness is automatic.
- This applies to the example of the generalized micromaser. We conclude that **the generalized micromaser without trapping states is always asymptotically complete**.

Purification

- An interesting technique used in rigorously proving these results is the following purification method.
  (Can it be used for solving other problems too?)

- GNS-constructions:

$$
\begin{array}{ccc}
(\mathcal{A}, \varphi) & \triangleright & (\mathcal{H}, \xi_\varphi) \\
(\mathcal{C}, \psi) & \triangleright & (\mathcal{K}, \eta_\psi)
\end{array}
$$

$J : (\mathcal{A}, \varphi) \to (\mathcal{A} \otimes \mathcal{C}, \varphi \otimes \psi)$ stationary $\quad \triangleright \quad v : \mathcal{H} \to \mathcal{H} \otimes \mathcal{K}$ isometry

$$Z' : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}),\ x \mapsto v^* \, x \otimes \mathbb{1} \, v$$

is a unital completely positive map
on all bounded operators on the GNS-space

Extended transition

- We call the CP-map Z' the **dual extended transition operator**. In fact it extends the dual transition operator $T'_\varphi : \mathcal{A}' \to \mathcal{A}'$ for the commutant of the stationary Markov process.
- Important property: While the original transition operator T_ψ has the invariant faithful state φ the dual extended transition operator Z' has an invariant vector state, namely $|\xi_\varphi\rangle\langle\xi_\varphi|$, with the corresponding cyclic vector ξ_φ.

  Advantage: Correlations involving a vector state are easier to handle than correlations involving a mixed state.
  This makes the Z'-technique so useful.

Asymptotic Completeness and Extended Transition

- In fact, in our context the nice result is

  Theorem. *TFAE*
  1. *J is asymptotically complete.*
  2. *Z' is ergodic (no non-trivial fixed points)*
  3. *Z' is mixing:*

  $$(Z')^n(x) \to \langle \xi_\varphi, x\, \xi_\varphi \rangle \, \mathbb{1} \quad \textit{for all } x \textit{ and } n \to \infty$$

- Definitely not true if we replace Z' by the original transition operator T_φ. In fact, transition operators T can have different extensions in this sense, some ergodic and some not. Z' contains additional information about the interaction.
- All the relevant information is in the spectrum of Z'. This can be a useful criterion. For example if $\mathcal{A}$ is finite-dimensional then also Z' is a finite-dimensional object while the statements about universal preparability and asymptotic completeness used earlier all involve infinite dimension.

Tightness

- If $\mathcal{A}$ is infinite dimensional it is necessary to speak about tightness. Our setting allows to develop this notion.

  Tightness is a concept from classical probabilty theory. A family of probability measures on a Borel σ-algebra is **tight** if for all $\epsilon > 0$ there is a compact set such that the probability of its complement is at most ϵ for all members of the family. For discrete spaces we can talk of finite instead of compact sets.

  This is technically essential in dealing with limits: In fact a well-known theorem of **Prokhorov** says that for Polish spaces tightness is equivalent to weak relative compactness of the family of probability measures.

Noncommutative Prokhorov theorem

- An important technical step to achieve rigorous proofs for our results above is the following noncommutative version of Prokhorov's theorem:

  Theorem.
  A sequence of normal states on $\mathcal{B}(\mathcal{H})$ is tight (in the sense that for any $\epsilon > 0$ there exists a finite dimensional projection such that the expectations of its complement are all at most ϵ) if and only if it is relatively compact in the norm topology.

- Note that $\mathcal{B}(\mathcal{H})$ is an atomic von Neumann algebra and we should think of the result above as generalizing the classical Prokhorov theorem for discrete spaces. I am not aware of a worked out generalization of Prokhorov's theorem for general von Neumann algebras. This is one of the reasons why some of our main results are stated for $\mathcal{A} = \mathcal{B}(\mathcal{H})$.

Postscript: Tightness of a Transition

- Tightness of the transition J (appearing in one of our theorems) can be defined in terms of the dual transition operator Z':

  We require that for all normal states σ the sequences $\big(\sigma \circ (Z')^n\big)$ are tight.

  This is a necessary condition for asymptotic completeness because in this case Z' has an absorbing state.

Conclusion and Outlook

Final Remarks:

- I hope to have convinced you that the study of asymptotic completeness and of related criteria is relevant for the **development of a control theory for open quantum systems** and that the corresponding functional analytic and operator algebraic tools deserve further development.
- One example of further development: In control theory we have for controllability the dual notion of **observability**, meaning here that we can find out about the state of the system $\mathcal{A}$ by only measuring the environment (the $\mathcal{C}$-systems).

  We have a result that in the setting of stationary Markov processes, as introduced above, this corresponds to asymptotic completeness of the time-reversed system.

  Work on refinements of this is going on.
- **Thank you !**