Thoma's theorem about extremal characters as a noncommutative de Finetti theorem

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Theorem (Thoma 64,)

An **extremal character** of the group $\mathbb{S}_\infty$ is of the form

$$\chi(\pi) = \prod_{k=2}^{\infty} \left(\sum_{i=1}^{\infty} a_i^k + (-1)^{k-1} \sum_{j=1}^{\infty} b_j^k \right)^{m_k(\pi)} .$$

Here $m_k(\pi)$ is the number of disjoint k-cycles in the permutation π and the two sequences $(a_i)_{i=1}^{\infty}, (b_j)_{j=1}^{\infty}$ satisfy

$$a_1 \geq a_2 \geq \cdots \geq 0, \qquad b_1 \geq b_2 \geq \cdots \geq 0, \qquad \sum_{i=1}^{\infty} a_i + \sum_{j=1}^{\infty} b_j \leq 1.$$

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Note: **Thoma multiplicativity** for disjoint cycle decomposition.

Noncommutative random variables

From quantum mechanics we learn:

noncommutative **random variables**

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selfadjoint operators + expectation functional ϕ (state)

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A sequence $x_1, x_2, \ldots$ of (noncommutative) random variables is

exchangeable

if joint moments are invariant under permutations of the variables:

$$\phi(x_{i_1} \ldots x_{i_n}) = \phi(x_{\pi(i_1)} \ldots x_{\pi(i_n)})$$

Claus Köstler's noncommutative de Finetti theorem

de Finetti: exchangeability $\Rightarrow$ conditional independence

Claus Köstler has shown (JFA 2010) that this idea also works for noncommutative random variables:

Theorem

Every exchangeable sequence x_1, x_2 of (noncommutative) random variables is **conditionally independent** over the **tail algebra**

$$\mathcal{A}^{\text{tail}} := \bigcap_n \mathrm{vN}(x_n, x_{n+1}, \dots) .$$

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- vN means the generated von Neumann algebra
(weak closure in GNS-rep with respect to the state).
- Conditional independence:
factorization of the state-preserving conditional expectation E onto the tail algebra $\mathcal{A}^{\text{tail}}$:

$$E(xx') = E(x)E(x') \text{ if } x \in vN(x_i, i \in I), x' \in vN(x_j, j \in J),\ I \cap J = \emptyset$$

Characters and traces

Every character χ of $\mathbb{S}_\infty$ gives rise to a
unitary representation

$$\pi \colon \mathbb{S}_\infty \to \mathcal{U}(\mathcal{A}), \qquad \text{with } \mathcal{A} = \mathrm{vN}(\pi(\mathbb{S}_\infty))$$

such that there is a tracial state tr on $\mathcal{A}$.

The converse is also true.

Notation: omit π from now on $\quad \Rightarrow \mathrm{tr} = \chi$ on $\mathbb{S}_\infty$

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Reformulation of Thoma's problem:
Characterize **factorial tracial states** on the group algebra of $\mathbb{S}_\infty$.

Noncommutative probability approach

Find a natural exchangeable sequence of noncommutative random variables (=selfadjoint operators) which generate $\mathcal{A} = \mathrm{vN}(\mathbb{S}_\infty)$

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First try: the **Coxeter generators** $\sigma_i = (i-1, i)$ $(i \in \mathbb{N})$.
Not exchangeable (in general).

Star generators are exchangeable

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They are exchangeable:

$$\begin{aligned}
& \chi(\gamma_{i_1}\gamma_{i_2}\dots\gamma_{i_n}) \\
= \ & \chi(\pi\gamma_{i_1}\gamma_{i_2}\dots\gamma_{i_n}\pi^{-1}) \\
= \ & \chi\big((\pi\gamma_{i_1}\pi^{-1})(\pi\gamma_{i_2}\pi^{-1})\dots(\pi\gamma_{i_n}\pi^{-1})\big) \\
= \ & \chi(\gamma_{\pi(i_1)}\gamma_{\pi(i_2)}\dots\gamma_{\pi(i_n)})
\end{aligned}$$

where π is any permutation with $\pi(0) = 0$.

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where π is any permutation with $\pi(0) = 0$.

So this is a nice example of a

noncommutative exchangeable sequence of coin tosses.

Star generators and cycles

Another useful elementary fact:

A k-**cycle** $\tau = (n_1\, n_2\, n_3 \ldots n_k) \in \mathbb{S}_\infty$ can be written in the star generators γ_i as

$$\tau = \gamma_{n_1}\gamma_{n_2}\gamma_{n_3}\cdots\gamma_{n_{k-1}}\gamma_{n_k}\gamma_{n_1},$$

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And now we get
from the noncommutative de Finetti theorem:

$$E(\tau\,\tau') = E(\tau)E(\tau')$$

τ and τ' are disjoint cycles
E is the conditional expectation onto the tail algebra of the sequence $\gamma_1, \gamma_2 \ldots$ of star generators.

Evaluation of the conditional expectation

What is $E(\tau)$ for a k-cycle τ ?

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Let us restrict to the case of an extremal character (factor trace) from now on.

Key observation: Let τ be a k-cycle. Then

$$E(\tau) = \begin{cases} A_0^{k-1} & \text{if } \tau(0) \neq 0 \\ \mathrm{tr}(A_0^{k-1})\, \mathbb{1} & \text{if } \tau(0) = 0 \end{cases}$$

Thoma multiplicativity

For $\pi \in \mathbb{S}_\infty$ let

$$\pi = \tau_1 \dots \tau_n$$

be the disjoint cycle decomposition.

At most one of the cycles, say τ_1, moves the point 0.

$$\begin{aligned}
\Rightarrow \quad \chi(\pi) &= \chi(\tau_1 \dots \tau_n) \\
&= \operatorname{tr}(E(\tau_1 \dots \tau_n)) \\
&= \operatorname{tr}(E(\tau_1)E(\tau_2)\dots E(\tau_n)) \\
&= \operatorname{tr}(A_0^{k_1-1}\operatorname{tr}(A_0^{k_2-1})\dots \operatorname{tr}(A_0^{k_n-1})) \\
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&= \chi(\tau_1)\chi(\tau_2) \dots \chi(\tau_n)
\end{aligned}$$

which means we have proved **Thoma multiplicativity**

as a consequence of the noncommutative de Finetti theorem.

Spectral Theory

It also follows that the tail algebra for the sequence $\gamma_1, \gamma_2, \dots$ is generated by the selfadjoint contraction A_0:

$$\mathcal{A}^{\mathrm{tail}} = \mathrm{vN}(A_0)$$

So $\mathcal{A}^{\mathrm{tail}}$ is commutative.

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$$\mathrm{tr}(\tau) = \mathrm{tr}(A_0^{k-1}) = \int_{-1}^{1} t^{k-1}\, d\mu\ ,$$

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Noncommutative independence also helps to prove that.

But that is another talk!

References

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