QUANTUM STOCHASTIC LIE-TROTTER PRODUCT FORMULA (two anniversaries)

Martin Lindsay

Lancaster University, U.K.

(joint work with Kalyan Sinha,

Jawaharlal Nehru Center for Advanced Scientific Research, Bangalore)

STOCHASTIC PROCESSES AT THE QUANTUM LEVEL

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Classical Product Formulae

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Remark.
The convergence is uniform on compact intervals.

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The *linear independence* and *totality* of the exponential vectors also facilitates the definition of operators on Fock spaces.

Quantum Stochastics

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Theorem. (*Parthasarathy-Sunder, Skeide*).
If T is total in k and contains 0 then $\big\{\varepsilon(f) : f \in \mathbb{S}'_{\mathsf{T}}\big\}$ is total in $\mathcal{F}_{\mathsf{k}}$.

Quantum Stochastic Processes

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Here $\mathcal{S}$ denotes the Hitsuda-Skorohod integral.

Remark.
These are all *unbounded* processes.

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where $(\sigma_s)_{s \geqslant 0}$ is the right shift on $B(\mathcal{F})$ (the *CCR flow with index* k) ampliated by $\mathrm{id}_{B(\mathfrak{h})}$.

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- The expectation semigroup of this cocycle has generator $-\frac{1}{2}H^2$.

Quantum Stochastic Generators

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The Problem

Let $V^{(1)}$ and $V^{(2)}$ be Markov-regular QS contraction cocycles on $\mathfrak{h}$ with noise dimension spaces k_1 and k_2 respectively, and quantum stochastic generators $F_1 := \begin{bmatrix} K_1 & M_1 \\ L_1 & W_1 - I_1 \end{bmatrix} \in B\big(\mathfrak{h} \oplus (\mathfrak{h} \otimes \mathsf{k}_1)\big)$ and $F_2 := \begin{bmatrix} K_2 & M_2 \\ L_2 & W_2 - I_2 \end{bmatrix} \in B\big(\mathfrak{h} \oplus (\mathfrak{h} \otimes \mathsf{k}_2)\big)$.

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Remark.

- In case $\mathsf{k} = \{0\}$, this reduces to the Lie-Trotter problem.

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where $t_0 = 0$, $t_{n+1} = t$ and

$$\big\{t_1 < \cdots < t_n\big\} \subset \mathbb{D}_+$$

is the (possibly empty) union of the sets of points of discontinuity of f and g in $]0,t[$.

Let $V^{(1)}$, $V^{(2)}$ and V be Markov-regular QS contraction cocycles on $\mathfrak{h}$ with noise dimension spaces k_1, k_2 and $\mathsf{k} = \mathsf{k}_1 \oplus \mathsf{k}_2$ respectively, and quantum stochastic generators $F_1 := \left[\begin{smallmatrix} K_1 & M_1 \\ L_1 & W_1 - I_1 \end{smallmatrix}\right]$, $F_2 := \left[\begin{smallmatrix} K_2 & M_2 \\ L_2 & W_2 - I_2 \end{smallmatrix}\right]$ and

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$$t_k^n := 2^{-n}\big([2^n t] + k\big) \text{ for } n \in \mathbb{N} \text{ and } k \in \mathbb{Z} \text{ s.t. } k \geqslant -[2^n t].$$

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Thus

$$t_0^n \leqslant t_0^{n+1} \leqslant t \leqslant t_1^{n+1} \leqslant t_1^n \text{ and } |t_{k+1}^n - t_k^n| = 2^{-n}.$$

The Solution: Lemma

Define

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Lemma.
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$$V^{(1,2)}_n(t) := \Big(V^{(1,2)}_{0,2^{-n}}\ V^{(1,2)}_{2^{-n},2.2^{-n}} \cdots V^{(1,2)}_{t^n_{-1},t^n_0}\Big)\ V^{(1,2)}_{t^n_0,t} \quad (t \in \mathbb{R}_+).$$

Lemma.
Let $f, g \in \mathbb{S}'$, and let $N \in \mathbb{N}$ be sufficiently large that f and g are constant on intervals of the form $[2^{-N}j, 2^{-N}(j+1)[$. Then, for all $n \geqslant N$,

$$\big(\mathrm{id}_{B(\mathfrak{h})} \overline{\otimes} \omega_{\varepsilon(f_{[0,t[}),\varepsilon(g_{[0,t[})}\big)\big(V^{(1,2)}_n(t)\big) =$$

$$\Big[{}^{(1,2)}P^{f(0),g(0)}_{2^{-n}}\Big]\ \Big[{}^{(1,2)}P^{f(2^{-n}),g(2^{-n})}_{2^{-n}}\Big] \cdots \Big[{}^{(1,2)}P^{f(t^n_{-1}),g(t^n_{-1})}_{2^{-n}}\Big]\ \Big[{}^{(1,2)}P^{f(t^n_0),g(t^n_0)}_{t-t^n_0}\Big]$$

where

$${}^{(1,2)}P^{(c,d)}_t := \Big[{}^{(1)}P^{(c^1,d^1)}_t\Big]\ \Big[{}^{(2)}P^{(c^2,d^2)}_t\Big],$$

for $c = \binom{c^1}{c^2}$ and $d = \binom{d^1}{d^2}$ in $\mathsf{k} = \mathsf{k}_1 \oplus \mathsf{k}_2$.
Here ${}^{(1)}P$ and ${}^{(2)}P$ denote the associated semigroups of $V^{(1)}$ and $V^{(2)}$.

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we have

$$\Big(V_{0,2^{-n}}^{(1,2)}\, V_{2^{-n},2.2^{-n}}^{(1,2)} \cdots V_{t_{-1}^n,t_0^n}^{(1,2)}\Big)\, V_{t_0^n,t}^{(1,2)} \longrightarrow V_t \text{ (W.O.T.)}$$

as $n \longrightarrow \infty$ $(t \in \mathbb{R}_+)$.

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- Again the convergence is in a hybrid topology in general (now S.O.T-W.O.T.), strongly converging when V is isometric.
- There are corresponding QS Product Formulae for QS mapping cocycles on operator spaces and QS flows on C^*-algebras.