Noncommutative Markov Chains and Multianalytic Operators

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Stochastic Processes at the Quantum Level

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Linear systems

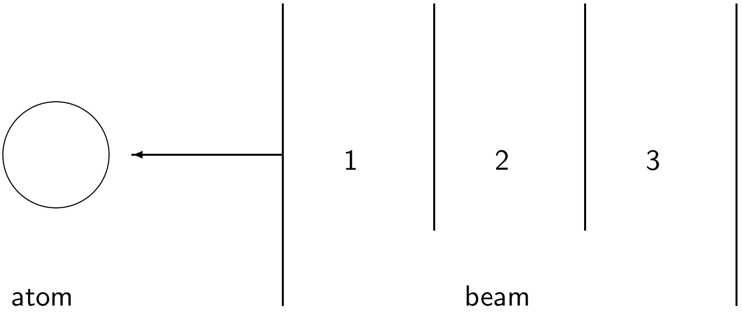
$$\begin{aligned} x_{n+1} &= A\,x_n + B\,u_n \\ y_n &= C\,x_n + D\,u_n \end{aligned}$$

Given x_0 and $(u_n)_{n\in\mathbb{N}_0}$ we can use these equations to compute $(x_n)_{n\in\mathbb{N}_0}$ and $(y_n)_{n\in\mathbb{N}_0}$ recursively.

output	internal state	input
$\longleftarrow$		$\longleftarrow$
$(y_n)_{n\in\mathbb{N}_0}$	$(x_n)_{n\in\mathbb{N}_0}$	$(u_n)_{n\in\mathbb{N}_0}$

Toy model 1

We want to discuss a new approach to describe quantum mechanical systems in the language of linear systems.



Toy model 2

Given

- three Hilbert spaces $\mathcal{H}$, $\mathcal{K}$, $\mathcal{P}$
- a unitary operator $U : \mathcal{H} \otimes \mathcal{K} \to \mathcal{H} \otimes \mathcal{P}$ $(U^* U = UU^* = \mathbb{1})$
- unit vectors $\Omega^{\mathcal{H}} \in \mathcal{H}$, $\Omega^{\mathcal{K}} \in \mathcal{K}$, $\Omega^{\mathcal{P}} \in \mathcal{P}$ such that

$$U \left(\Omega^{\mathcal{H}} \otimes \Omega^{\mathcal{K}} \right) = \Omega^{\mathcal{H}} \otimes \Omega^{\mathcal{P}}$$

we call U an **interaction** with **vacuum vectors** $\Omega^{\mathcal{H}}, \Omega^{\mathcal{K}}, \Omega^{\mathcal{P}}$.

Model of repeated interaction 1

Infinite Hilbert space tensor products

$$\mathcal{K}_\infty := \bigotimes_{\ell=1}^{\infty} \mathcal{K}_\ell \qquad \mathcal{K}_\ell \simeq \mathcal{K}$$

$$\mathcal{P}_\infty := \bigotimes_{\ell=1}^{\infty} \mathcal{P}_\ell \qquad \mathcal{P}_\ell \simeq \mathcal{P}$$

along unit vectors $\Omega_\infty^{\mathcal{K}} = \bigotimes_1^\infty \Omega^{\mathcal{K}}$ and $\Omega_\infty^{\mathcal{P}} = \bigotimes_1^\infty \Omega^{\mathcal{P}}$.

natural embeddings

$$\mathcal{H} \simeq \mathcal{H} \otimes \Omega_\infty^{\mathcal{K}} \subset \mathcal{H} \otimes \mathcal{K}_\infty \supset \Omega^{\mathcal{H}} \otimes \mathcal{K}_\infty \simeq \mathcal{K}_\infty.$$

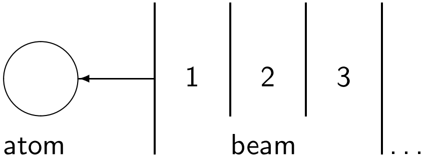
Model of repeated interaction 2

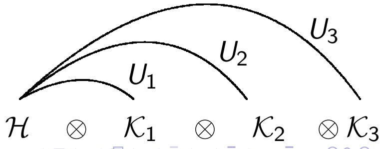
We can now define repeated interactions. For $\ell \in \mathbb{N}$ let

$$U_\ell : \mathcal{H} \otimes \mathcal{K}_\infty \to \mathcal{H} \otimes \mathcal{K}_{[1,\ell-1]} \otimes \mathcal{P}_\ell \otimes \mathcal{K}_{[\ell+1,\infty)}$$

be the unitary operator which is equal to U on $\mathcal{H} \otimes \mathcal{K}_\ell$ and which acts identically on the other factors of the tensor product. Then the repeated interaction up to time $n \in \mathbb{N}$ is defined by

$$U(n) := U_n \ldots U_1 : \mathcal{H} \otimes \mathcal{K}_\infty \to \mathcal{H} \otimes \mathcal{P}_{[1,n]} \otimes \mathcal{K}_{[n+1,\infty)}$$





Markovian nature of the model

We can think of our model as a **noncommutative Markov chain** or, from a physicist's point of view, as a Markovian approximation of a repeated atom-field interaction.

Change of an observable $X \in \mathcal{B}(\mathcal{H})$ until time n compressed to $\mathcal{H}$:

$$Z_n(X) = P_{\mathcal{H}}\, U(n)^* \; X \otimes 1 \; U(n)|_{\mathcal{H}}.$$

For ONB (ϵ_j) of the Hilbert space $\mathcal{P}$ and for $\xi \in \mathcal{H}$ write

$$U(\xi \otimes \Omega^{\mathcal{K}}) = \sum_j A_j \xi \otimes \epsilon_j$$

with operators $A_j \in \mathcal{B}(\mathcal{H})$. Then

$$Z_n(X) = \sum_{j_1, j_2, \ldots, j_n} A_{j_1}^* \ldots A_{j_n}^* \, X \, A_{j_n} \ldots A_{j_1} = Z^n(X),$$

where $Z = \sum_j A_j^* \cdot A_j : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is a noncommutative **transition operator**. This semigroup property is one of the basic features of Markovianity.

Some concepts from multivariate operator theory

$T_1, \dots, T_d \in \mathcal{B}(\mathcal{L})$ for a Hilbert space $\mathcal{L}$ $\quad (d = \infty$ allowed$)$

$\underline{T} = (T_1, \dots, T_d)$ is called a **row contraction** if it is contractive as an operator from $\bigoplus_1^d \mathcal{L}$ to $\mathcal{L}$ or, equivalently, if $\sum_1^d T_j T_j^* \leq 1$.

$\underline{T} = (T_1, \dots, T_d)$ is called a **row isometry** if it is isometric as an operator from $\bigoplus_1^d \mathcal{L}$ to $\mathcal{L}$ or, equivalently, if the T_j are isometries with orthogonal ranges.

A row isometry $\underline{T} = (T_1, \dots, T_d)$ is called a **row shift** if there exists a subspace $\mathcal{E}$ of $\mathcal{L}$ (the wandering subspace) such that $\mathcal{L} = \bigoplus_{\alpha \in F_d^+} T_\alpha \mathcal{E} \quad (F_d^+$ free semigroup with generators $1, \dots, d)$

Some concepts from multivariate operator theory 2

An **outgoing Cuntz scattering system** is a collection

$$(\mathcal{L}, \underline{V} = (V_1, \dots, V_d), \mathcal{G}_*^+, \mathcal{G})$$

where $\underline{V}$ is a row isometry on the Hilbert space $\mathcal{L}$ and $\mathcal{G}_*^+$ and $\mathcal{G}$ are subspaces of $\mathcal{L}$ such that

1. $\mathcal{G}_*^+$ is the smallest $\underline{V}$-invariant subspace containing

$$\mathcal{E}_* := \mathcal{L} \ominus span_{j=1,\dots,d} \ V_j \mathcal{L} \ ,$$

   thus $\underline{V}|_{\mathcal{G}_*^+}$ is a row shift and $\mathcal{G}_*^+ = \bigoplus_{\alpha \in F_d^+} V_\alpha \mathcal{E}_*$
2. $\underline{V}|_{\mathcal{G}}$ is a row shift, thus $\mathcal{G} = \bigoplus_{\alpha \in F_d^+} V_\alpha \mathcal{E}$ with

$$\mathcal{E} := \mathcal{G} \ominus span_{j=1,\dots,d} \ V_j \mathcal{G}.$$

Outgoing Cuntz scattering system from interaction model

Theorem:

Let U be an interaction with vacuum vectors $\Omega^{\mathcal{H}}$, $\Omega^{\mathcal{K}}$, $\Omega^{\mathcal{P}}$. Then we have an outgoing Cuntz scattering system

$$(\mathcal{H} \otimes \mathcal{K}_\infty)^o,\ \underline{V} = (V_1, \dots, V_d),\ \mathcal{G}_*^+,\ \mathcal{G}$$

where

$$(\mathcal{H} \otimes \mathcal{K}_\infty)^o := (\mathcal{H} \otimes \mathcal{K}_\infty) \ominus \mathbb{C}(\Omega^{\mathcal{H}} \otimes \Omega^{\mathcal{K}}_\infty)$$

$$V_j(\xi \otimes \eta) := U^*(\xi \otimes \epsilon_j) \otimes \eta \in (\mathcal{H} \otimes \mathcal{K}_1) \otimes \mathcal{K}_{[2,\infty)}$$

$$\mathcal{E}_* = W^* \mathcal{Y} \subset \mathcal{H} \otimes \mathcal{K}_1, \quad \mathcal{G}_*^+ = \bigoplus_{\alpha \in F_d^+} V_\alpha \mathcal{E}_*$$

with $W^* = sot-\lim_{n\to\infty} U(n)^*|_{\mathcal{P}^o_\infty}$, $\quad \mathcal{Y} := \Omega^{\mathcal{H}} \otimes (\Omega_1^{\mathcal{P}})^\perp \otimes \Omega_{[2,\infty)} \subset \mathcal{P}^o_\infty$

$$\mathcal{E} := \mathcal{H} \otimes (\Omega_1^{\mathcal{K}})^\perp \otimes \Omega^{\mathcal{K}}_{[2,\infty)}, \quad \mathcal{G} = \bigoplus_{\alpha \in F_d^+} V_\alpha \mathcal{E}.$$

F_d^+-linear systems 1

- **input space** $\mathcal{U} := \mathcal{E} = \mathcal{H} \otimes (\Omega_1^{\mathcal{K}})^{\perp} \otimes \Omega_{[2,\infty)}^{\mathcal{K}} \quad \subset (\mathcal{H} \otimes \mathcal{K}_\infty)^o$,
- **output space** $\mathcal{Y} := (\Omega_1^{\mathcal{P}})^{\perp} \otimes \Omega_{[2,\infty)}^{\mathcal{P}} \quad \subset (\mathcal{P}_\infty)^o$

With $H \otimes \mathcal{K} = \mathcal{H} \oplus \mathcal{U}$ the interaction U maps $\mathcal{H} \oplus \mathcal{U}$ onto $\mathcal{H} \otimes \mathcal{P}$ which contains $\mathcal{Y}$ (identifying $\mathcal{P}$ and $\mathcal{P}_1$). Hence for $j = 1, \ldots, d$ we can define

$$A_j : \mathcal{H} \to \mathcal{H}, \quad B_j : \mathcal{U} \to \mathcal{H}, \quad C : \mathcal{H} \to \mathcal{Y}, \quad D : \mathcal{U} \to \mathcal{Y}$$

$$\begin{aligned} U(\xi \oplus \eta) &=: \sum_{j=1}^{d} \left(A_j \xi + B_j \eta\right) \otimes \epsilon_j \\ P_{\mathcal{Y}}\, U(\xi \oplus \eta) &=: C\xi + D\eta, \end{aligned}$$

with $\xi \in \mathcal{H}$, $\eta \in \mathcal{U}$ and $\left(\epsilon_j\right)_{j=1}^{d}$ ONB of $\mathcal{P}$ and $P_{\mathcal{Y}}$ proj. onto $\mathcal{Y}$

F_d^+-linear systems 2

Further we define the **colligation**

$$\mathcal{C}_U := \begin{pmatrix} A_1 & B_1 \\ \vdots & \vdots \\ A_d & B_d \\ C & D \end{pmatrix} : \quad \mathcal{H} \oplus \mathcal{U} \to \bigoplus_{j=1}^{d} \mathcal{H} \oplus \mathcal{Y}$$

As usual, the colligation $\mathcal{C}_U$ gives rise to a F_d^+-**linear system** Σ_U (noncommutative Fornasini-Marchesini system)

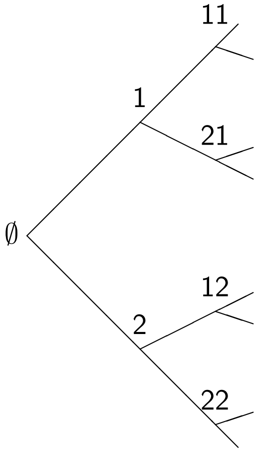
$$\begin{aligned} x(j\alpha) &= A_j\, x(\alpha) + B_j\, u(\alpha) \\ y(\alpha) &= C\, x(\alpha) + D\, u(\alpha), \end{aligned}$$

where $j = 1, \ldots, d$, further α, $j\alpha$ (concatenation) are words in F_d^+ and

$$x : F_d^+ \to \mathcal{H}, \quad u : F_d^+ \to \mathcal{U}, \quad y : F_d^+ \to \mathcal{Y}.$$

F_d^+-linear systems 3

Given $x(\emptyset)$ and u we can use Σ_U to compute x and y recursively.



$\ldots$

dyadic tree for $d = 2$

Transfer function 1

A very elegant way to encode all the information about the evolution of an F_d^+-linear system into a single mathematical object is the use of a transfer function. For this we define the 'Fourier transform' of x as

$$\hat{x}(z) = \sum_{\alpha \in F_d^+} x(\alpha) z^\alpha,$$

where $z^\alpha = z_{\alpha_n} \dots z_{\alpha_1}$ if $\alpha = \alpha_n \dots \alpha_1 \in F_d^+$ and $z = (z_1, \dots, z_d)$ is a d-tuple of formal non-commuting indeterminates. Similarly $\hat{u}(z) = \sum_{\alpha \in F_d^+} u(\alpha) z^\alpha$ and $\hat{y}(z) = \sum_{\alpha \in F_d^+} y(\alpha) z^\alpha$.
For $x(\emptyset) = 0$ we have the **input-output relation**

$$\hat{y}(z) = \Theta_U(z)\, \hat{u}(z)$$

where

$$\Theta_U(z) := \sum_{\alpha \in F_d^+} \Theta_U^{(\alpha)} z^\alpha := D + C \sum_{\substack{\beta \in F_d^+ \\ j = 1, \dots, d}} A_\beta B_j z^{\beta j}$$

Transfer function 2

We call the formal non-commutative power series $\Theta_U(z) := \sum_{\alpha \in F_d^+} \Theta_U^{(\alpha)} z^\alpha$ the **transfer function** associated to the interaction U. The 'Taylor coefficients' $\Theta_U^{(\alpha)}$ are operators from $\mathcal{U}$ to $\mathcal{Y}$.

Now we want to proceed from formal power series to operators between Hilbert spaces.

Theorem

The input-output relation

$$\hat{y}(z) = \Theta_U(z)\, \hat{u}(z)$$

corresponds to a contraction

$$M_{\Theta_U} : \ell^2(F_d^+, \mathcal{U}) \to \ell^2(F_d^+, \mathcal{Y})$$

which (with $x(\emptyset) = 0$) maps an input sequence u to the corresponding output sequence y.

Transfer function 3

The operator M_{Θ_U} has the property that it intertwines with right translation, i.e., for all $j = 1, \ldots, d$

$$M_{\Theta_U}\Big(\sum_{\alpha \in F_d^+} x(\alpha) z^\alpha \, z^j \Big) \; = \; M_{\Theta_U}\Big(\sum_{\alpha \in F_d^+} x(\alpha) z^\alpha \Big) \, z^j \; .$$

Such operators have been called **analytic intertwining operators** or **multianalytic operators**: there are analogies to the theory of multiplication operators by analytic functions on Hardy spaces. The non-commutative power series Θ_U is called the **symbol** of M_{Θ_U}.

It was one of the motivations for this work to make this theory available for the study of interaction models and non-commutative Markov chains. Note that because M_{Θ_U} is a contraction the transfer function Θ_U belongs to the socalled **non-commutative Schur class** $S_{nc,d}(\mathcal{U}, \mathcal{Y})$.

Physical interpretation 1

We may think of $\mathcal{H}$ as the (quantum mechanical) Hilbert space of an atom, $\mathcal{K}_\ell$ as the Hilbert space of a part of a light beam or field which interacts with the atom at time ℓ.

We think of $\Omega^{\mathcal{H}}$ as a vacuum state of the atom and of $\Omega^{\mathcal{K}} = \Omega^{\mathcal{P}}$ in $\mathcal{K} = \mathcal{P}$ as a state indicating that no photon is present. Then

$$\eta \in \mathcal{U} = \mathcal{H} \otimes (\Omega_1^{\mathcal{K}})^{\perp} \otimes \Omega_{[2,\infty)}^{\mathcal{K}} \subset \mathcal{H} \otimes \mathcal{K}_\infty$$

represents a vector state with

- photons arriving at time 1 and stimulating an interaction between the atom and the field,
- but no further photons arriving at later times.
- Nevertheless it may happen that some activity (emission) is induced which goes on for a longer period.

Physical interpretation 2

The orthogonal projection P_α onto

$$\epsilon_{\alpha_1} \otimes \ldots \otimes \epsilon_{\alpha_{n-1}} \otimes (\Omega_n^{\mathcal{P}})^\perp \otimes \Omega_{[n+1,\infty)},$$

corresponds to the following event:

- We measure data $\alpha_1, \ldots, \alpha_{n-1}$ at times $1, \ldots, n-1$ in the field, finally there is a last detection of photons corresponding to $(\Omega_n^{\mathcal{P}})^\perp$ at time n, nothing happens after time n.
- This experimental record is obtained by measuring (at times indexed by the positive integers) an observable $Y \in \mathcal{B}(\mathcal{P})$ with eigenvectors $\epsilon_1, \ldots, \epsilon_d$. Such lists of data have been used for indirect measurements of an atom, for quantum filtering and for updating protocols such as quantum trajectories.

Physical interpretation 3

We can obtain the formula

$$P_\alpha \, U(n)\eta \,=\, \Theta_U^{(\alpha)} \eta$$

According to the usual probabilistic interpretation of quantum mechanics this means for example that

$$\pi_\alpha := \| \, \Theta_U^{(\alpha)} \eta \, \|^2$$

is the probability for the event described by P_α if we start in the state η at time 0.

- Actually the transfer function also keeps track of the complex amplitudes and contains additional coherent information.

Conclusion: We can think of the transfer function Θ_U as a convenient way to assemble such data into a single mathematical object.

Outlook

- The control theoretic concept of 'observability' for our model is closely related to an operator-algebraic scattering theory for noncommutative Markov chains.
- Developing 'noncommutative analyticity' with applications to open quantum systems and quantum control seems to be a promising idea.

For the contents of this talk and for references see

Rolf Gohm, Non-Commutative Markov Chains and Multi-Analytic Operators, arXiv:0902.3445