Quantum Feynman–Kac perturbations

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Stochastic Processes at the Quantum Level

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1. Classical flows

Fix a von Neumann algebra $\mathcal{A} \subseteq \mathcal{B}(\mathsf{h})$. Let $\alpha = (\alpha_t : t \in \mathbb{R})$ be an ultraweakly continuous group of $*$-automorphisms of $\mathcal{A}$ and let δ be its ultraweak generator.

Gaussian subordination may be used to construct an ultraweakly continuous semigroup P^0 on $\mathcal{A}$ with ultraweak pre-generator $\frac{1}{2}\delta^2$.

If $(B_t : t \geqslant 0)$ is a standard Brownian motion and $\mathbb{P}$ is Wiener measure then

$$j_t : \mathcal{A} \to \mathcal{A} \,\overline{\otimes}\, \mathcal{B}(L^2(\mathbb{P}));\ a \mapsto \alpha_{B_t}(a) \otimes I_{L^2(\mathbb{P})} \qquad (t \geqslant 0)$$

is a $*$-homomorphism such that

$$j_t(x) = x + \int_0^t j_s(\delta(x))\,\mathrm{d}B_s + \frac{1}{2}\int_0^t j_s(\delta^2(x))\,\mathrm{d}s \qquad (\star)$$

strongly on $L^2(\mathbb{P}; \mathsf{h})$, for all $x \in \mathrm{dom}(\delta^2)$.

Setting

$$P_t^0(a)u := \mathbb{E}[j_t(a)u] \qquad (a \in \mathcal{A},\ u \in \mathsf{h} \subseteq L^2(\mathbb{P}; \mathsf{h}))$$

defines a semigroup $(P_t^0 : t \geqslant 0)$ of completely positive contractions on $\mathcal{A}$ with generator as desired.

1. Classical flows

Cocycle structure

The process j is adapted in the following sense: for all $t \geqslant 0$ and $a \in \mathcal{A}$,

$$j_t(a) = j_{t]}(a) \otimes I_{L^2(\mathbb{P}_{[t})}, \qquad \text{where } j_{t]}(a) \in \mathcal{A} \overline{\otimes} \mathcal{B}(L^2(\mathbb{P}_{t]})).$$

Let

$$\widehat{\jmath}_t := j_{t]} \overline{\otimes} I_{\mathcal{B}(L^2(\mathbb{P}_{[t}))} : \ \mathcal{A} \overline{\otimes} \mathcal{B}(L^2(\mathbb{P}_{[t})) \to \mathcal{A} \overline{\otimes} \mathcal{B}(L^2(\mathbb{P})).$$

If

$$\sigma_t : \mathcal{B}(L^2(\mathbb{P}; \mathsf{h})) \to \mathcal{B}(L^2(\mathbb{P}_{[t}; \mathsf{h})) \qquad (t \geqslant 0)$$

is the unital $*$-isomorphism given by the natural shift on the path space then

$$j_{s+t} = \widehat{\jmath}_s \circ \sigma_s \circ j_t \qquad \text{for all } s, t \geqslant 0,$$

so j is a cocycle for the shift semigroup σ.

Furthermore, if

$$J_t := \widehat{\jmath}_t \circ \sigma_t|_{\mathcal{A} \overline{\otimes} \mathcal{B}(L^2(\mathbb{P}))}$$

then $J = (J_t : t \geqslant 0)$ is a semigroup.

1. Classical flows

L–S perturbation

Given $b = b^* \in \mathcal{A}$, Lindsay and Sinha proved the existence of an adapted, operator-valued process m^b such that

$$m_t^b = I + \int_0^t j_s(b) m_s^b \, \mathrm{d}B_s \qquad (t \geqslant 0)$$

strongly on $L^2(\mathbb{P}; \mathsf{h})$.

If α is unitarily implemented, they showed that the *exponential martingale* m^b satisfies the J-cocycle identity

$$m_{s+t}^b = J_s(m_t^b) m_s^b$$

for all $s, t \geqslant 0$.

It follows that setting

$$P_t^b(a)u := \mathbb{E}[j_t(a) m_t^b u] \qquad (a \in \mathcal{A}, \ u \in \mathsf{h})$$

gives an ultraweakly continuous semigroup $(P_t^b : t \geqslant 0)$ with generator which extends

$$\tfrac{1}{2}\delta^2 + \rho_b \delta : \mathrm{dom}(\delta^2) \to \mathcal{A}; \ x \mapsto \tfrac{1}{2}\delta^2(x) + \delta(x) b.$$

1. Classical flows

B–P perturbation

Bahn and Park noted that such a L–S semigroup will not, in general, be positive or even real ($*$-preserving).

They investigated a more symmetric perturbation, using a J-cocycle n^b such that

$$n_t^b f = f + \int_0^t j_s(b)\mathbb{E}[n_s^b f \,|\, \mathcal{F}_s]\,\mathrm{d}B_s - \frac{1}{2}\int_0^t j_s(b^2)\mathbb{E}[n_s^b f \,|\, \mathcal{F}_s]\,\mathrm{d}s \qquad (\dagger)$$

for all $f \in L^2(\mathbb{P};\mathsf{h})$, where $(\mathcal{F}_t : t \geqslant 0)$ is the natural filtration of the Wiener process B.

In this case, letting

$$Q_t^b(a)u := \mathbb{E}[(n_t^b)^* j_t(a) n_t^b u] \qquad (a \in \mathcal{A},\ u \in \mathsf{h})$$

gives an ultraweakly continuous completely positive semigroup Q^b on $\mathcal{A}$, which is contractive if $b = b^*$ and whose generator extends

$$\tfrac{1}{2}\delta^2 + \lambda_b\delta + \rho_b\delta + \lambda_b\rho_b - \tfrac{1}{2}\lambda_{b^2} - \tfrac{1}{2}\rho_{b^2},$$

where $\lambda_c : a \mapsto ca$ and $\rho_c : a \mapsto ac$ are left and right-multiplication operators.

2. Quantum flows

To generalise, the process j constructed from α replaced with a quantum flow.

Recall that $L^2(\mathbb{P})$ is isomorphic to $\Gamma = \Gamma(L^2(\mathbb{R}_+))$, the Boson Fock space over $L^2(\mathbb{R}_+)$, and

$$\Gamma \cong \Gamma_{t]} \otimes \Gamma_{[t}$$

where

$$\Gamma_{t]} = \Gamma(L^2[0,t)) \cong L^2(\mathbb{P}_{t]}) \qquad \text{and} \qquad \Gamma_{[t} = \Gamma(L^2[t,\infty)) \cong L^2(\mathbb{P}_{[t}).$$

A *quantum flow* $j = (j_t : t \geqslant 0)$ is a family of $*$-homomorphisms

$$j_t : \mathcal{A} \to \mathcal{A} \overline{\otimes} \mathcal{B}(\Gamma)$$

which are

- vacuum adapted, so that

$$j_t(a) = j_{t]}(a) \otimes |\Omega_t\rangle\langle\Omega_t| \qquad \text{with} \qquad j_{t]}(a) \in \mathcal{A} \overline{\otimes} \mathcal{B}(\Gamma_{t]}),$$

  where $\Omega_t \in \Gamma_{[t}$ is the vacuum,

- such that $a \mapsto j_t(a)$ and $t \mapsto j_t(a)$ are ultraweakly continuous, and
- unital, in the sense that $j_{t]}(I) = I$.

2. Quantum flows

Moreover, j is required to satisfy the cocycle equation

$$j_{s+t} = \widehat{\jmath}_s \circ \sigma_s \circ j_t \qquad \text{for all } s, t \geqslant 0,$$

where

$$\widehat{\jmath}_t := j_{t]} \overline{\otimes} I_{\mathcal{B}(\Gamma_{[t})} : \ \mathcal{A} \overline{\otimes} \mathcal{B}(\Gamma_{[t}) \to \mathcal{A} \overline{\otimes} \mathcal{B}(\Gamma)$$

and

$$t \mapsto \sigma_t : \mathcal{B}(\mathsf{h} \otimes \Gamma) \to \mathcal{B}(\mathsf{h} \otimes \Gamma_{[t})$$

is the CCR flow.

2. Quantum flows

The flow j is assumed to satisfy the quantum stochastic differential equation

$$\mathrm{d}j_t(x) = j_t(\psi_\times^\times(x))\,\mathrm{d}\Lambda_t + j_t(\psi_\times^0(x))\,\mathrm{d}A_t + j_t(\psi_0^\times(x))\,\mathrm{d}A_t^\dagger + j_t(\psi_0^0(x))\,\mathrm{d}t \quad (\ddagger)$$

for all $x \in \mathcal{A}_0 \subseteq \mathcal{A}$, where the *structure maps*

$$\psi_\times^\times, \quad \psi_\times^0, \quad \psi_0^\times, \quad \psi_0^0 : \mathcal{A}_0 \to \mathcal{A}.$$

The QSDE ($\ddagger$) generalises the equation ($\star$), to which it reduces when

$$\mathcal{A}_0 = \mathrm{dom}(\delta^2), \qquad \psi_\times^\times = I_{\mathcal{A}_0}, \qquad \psi_\times^0 = \psi_0^\times = \delta|_{\mathcal{A}_0} \quad \text{and} \quad \psi_0^0 = \tfrac{1}{2}\delta^2.$$

(The gauge term $\psi_\times^\times$ is non-zero as j is vacuum adapted.)

The equation ($\ddagger$) implies that the flow j has Markov semigroup P^0 such that

$$\langle u, P_t^0(x)v\rangle = \langle u\,\Omega, j_t(x)v\,\Omega\rangle = \langle u, v\rangle + \int_0^t \langle u, j_s(\psi_0^0(x))v\rangle\,\mathrm{d}s \qquad (u, v \in \mathsf{h})$$

for all $t \geqslant 0$ and $x \in \mathcal{A}_0$, where $\Omega \in \Gamma$ is the vacuum.

Hence the generator of P^0 extends ψ_0^0.

2. Quantum flows

Previous authors (Evans and Hudson, Bradshaw, Das and Sinha) have examined perturbations of quantum flows given by conjugation with a unitary process.

This work focused on the situation where the structure maps of the flow j are elements of $\mathcal{B}(\mathcal{A})$, in which case the Markov semigroup is uniformly continuous.

If $h = h^* \in \mathcal{A}$ and $l \in \mathcal{A}$ then there exists a unitary process U such that

$$U_0 = I$$

and
$$\mathrm{d}U_t = j_t(-l^*)U_t \,\mathrm{d}A_t + j_t(l)U_t \,\mathrm{d}A_t^\dagger + j_t(-\mathrm{i}h - \tfrac{1}{2}l^*l)U_t \,\mathrm{d}t.$$

The process U is a J-cocycle and the Markov semigroup of the perturbed flow

$$(a \mapsto U_t^* j_t(a) U_t : t \geqslant 0)$$

has generator

$$\psi_0^0 + \rho_l \psi_\times^0 + \lambda_{l^*} \psi_0^\times + \rho_l \lambda_{l^*} \psi_\times^\times + \mathrm{i}[h, \cdot] - \tfrac{1}{2}\{l^* l, \cdot\},$$

where $[\cdot, \cdot]$ is the commutator and $\{\cdot, \cdot\}$ the anticommutator,

2. Quantum flows

Let $c = (c_0, c_\times) \in \mathcal{A} \times \mathcal{A}$. There exists a unique process M^c such that $M^c - I$ is vacuum adapted and satisfies the QSDE

$$\mathrm{d}(M^c - I)_t = j_t(c_0) M_t^c \,\mathrm{d}t + j_t(c_\times) M_t^c \,\mathrm{d}A_t^\dagger.$$

This is a generalisation of the B–P equation (†).

Furthermore, M^c is a J-cocycle: for all $s, t \geqslant 0$,

$$M_{s+t}^c = J_s(M_t^c) M_s^c.$$

To establish this, an identity of the form

$$\Big(\int_s^t F_r \,\mathrm{d}\Xi_r \Big) G_s = \int_s^t F_r G_s \,\mathrm{d}\Xi_r$$

is required, where $\Xi_r \in \{A_r^\dagger, r\}$.

This identity is simple to establish for these integrators, but does not hold in the vacuum-adapted setting for annihilation or gauge integrals.

2. Quantum flows

Let $c = (c_0, c_\times)$, $d = (d_0, d_\times) \in \mathcal{A} \times \mathcal{A}$.

There exists an ultraweakly continuous semigroup $P^{d,c}$ of completely bounded maps on $\mathcal{A}$ with

$$\langle u, P_t^{d,c}(a)v \rangle = \langle u\,\Omega, (M_t^d)^* j_t(a) M_t^c v\,\Omega \rangle \qquad (u, v \in \mathsf{h})$$

for all $t \geqslant 0$ and $a \in \mathcal{A}$. If $c = d$ then $P_t^{d,c}$ is completely positive for all $t \geqslant 0$.

The ultraweak generator of $P^{d,c}$ is an extension of

$$\psi_0^0 + \rho_{c_\times} \psi_\times^0 + \lambda_{d_\times^*} \psi_0^\times + \rho_{c_\times} \lambda_{d_\times^*} \psi_\times^\times + \rho_{c_0} + \lambda_{d_0^*}. \tag{§}$$

This class includes both the L–S and the B–P examples.

It also includes those obtained by unitary conjugation; the latter give a version of (§), subject to the constraints that $c_\times = d_\times = l$ and $c_0 = d_0 = -\mathrm{i}h - \frac{1}{2} l^* l$.

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